Solution
Week 1

Ex 1.1 & 1.2: 15, 22, 28, 32, 33

Ex 1.3: 13, 16, 25, 29, 34

Ex 1.4: 7, 17, 24, 26, 28

Ex 1.1 & 1.2: (15)

- Here we must place a, b, c, d in the the positions denoted by *x*: e <u>*x*</u> e <u>*x*</u> e <u>*x*</u> e <u>*x*</u> e.
- By the rule of product there are 4! ways to do this.

Ex 1.1 & 1.2: (22)

- Case1: The leading digit is 5: (6!)/(2!)
- Case2: The leading digit is 6: $(6!)/(2!)^2$
- Case3: The leading digit is 7: $(6!)/(2!)^2$
- In total there are [(6!)/(2!)][1+(1/2)+(1/2)] = 720 such position integers n.

Ex 1.1 & 1.2: (28)

a) The for loops for *i*, *j*, *k* are executed 12, 6, 8 times, respectively. The value of counter is
$0 + 12 \times 1 + 6 \times 2 + 8 \times 3 = 48$.

b) By the rule of sum.

Ex 1.1 & 1.2: (32)

a) For positive integers n, k where $n = 3k$, $n!/(3!)^k$ is the number of ways to arrange the n objects $x_1, x_1, x_1, x_2, x_2, x_2, \ldots, x_k, x_k, x_k$.
This must be an integer.

b) If n, k are positive integers with $n = mk$, then $n!/(m!)^k$ is an integer.

Ex 1.1 & 1.2: (33)

a) With 2 choices per question.
There are $2^{10} = 1024$ ways.

b) With 3 choices per question.
There are 3^{10} ways.

Ex 1.3: (13)

- The letters M,I,I,I,P,P,I can be arranged in [7!/(4!2!)] ways. Each arrangement provides 8 locations for placing the 4 nonconsecutive S's.
- Four of there locations can be selected in $\binom{s}{4}$ ways. Hence, total number of these arrangements is $\binom{s}{4}$[7!/(4!2!)].

Ex 1.3: (16)

a) 97
b) -5
c) 12
d) 0
e) 3

Ex 1.3: (25)

a) $\binom{4}{1,1,2} = 12$

b) $\binom{4}{0,1,1,2} = 12$

c) $\binom{4}{1,1,2}(2)(-1)(-1)^2 = -24$

d) $\binom{4}{1,1,2}(-2)(3)^2 = -216$

e) $\binom{8}{3,2,1,2}(2)^3(-1)^2(3)(-2)^2 = 161280$

Ex 1.3: (29)

* $n\binom{m+n}{m}$

$$= n\frac{(m+n)!}{m!\,n!}$$

$$= \frac{(m+n)!}{m!\,(n-1)!}$$

$$= (m+1)\,\frac{(m+n)!}{(m+1)(m!)(n-1)!}$$

$$= (m+1)\binom{m+n}{m+1}$$

Ex 1.3: (34)

a) **procedure** *Select2* (*i*,*j*: positive integers)
begin
    for *i* := 1 to 5 **do**
        for *j* := *i* + 1 to 6 **do**
            print (*i*,*j*)
end

b) **procedure** *Select3* (*i*,*j*,*k*: positive integers)
begin
    for *i* := 1 to 4 **do**
        for *j* := *i* + 1 to 5 **do**
            for *k* := *j* + 1 to 6 **do**
                print (*i*,*j*,*k*)
end

Ex 1.4: (7)

a) $\binom{4+32-1}{32} = \binom{35}{32}$
b) $\binom{4+28-1}{28} = \binom{31}{28}$
c) $\binom{4+8-1}{8} = \binom{11}{8}$
d) 1
e) Let $y_i = x_i+2$, $1 \le i \le 4$. The number of solutions to the given problem is then the same as the number of solutions to $y_1+y_2+y_3+y_4 = 40$, $0 \le y_i$, $1 \le i \le 4$. $\binom{4+40-1}{40} = \binom{43}{40}$.
f) $\binom{4+28-1}{28} - \binom{4+3-1}{3} = \binom{31}{28} - \binom{6}{3}$, where the term $\binom{6}{3}$ accounts for the solutions where $26 \le x_4$.

Ex 1.4: (17)

a) $\binom{5+12-1}{12} = \binom{16}{12}$

b) 5^{12}

Ex 1.4: (24)

a) **procedure** Selection1 (i,j: nonnegative integers)
begin
  for i := 0 to 10 **do**
    for j := 0 to 10 – i **do**
      print (i,j,10-i-j)
end

b) Let $y_i = x_i + 2 \geq 0$. It's equal to solve
$y_1+y_2+y_3+y_4 = 12$, where $y_i \geq 0$ for $1 \leq i \leq 4$.
The algorithm is like (a).

Ex 1.4: (26)

- Each such composition can be factored as k times a composition of m.
- Consequently, there are 2^{m-1} compositions of n, where $n = mk$ and each summand in a composition is a multiple of k.

Ex 1.4: (28.a)

A string of this type consists of x_1 1's followed by x_2 0's followed by x_3 1's followed by x_4 0's followed by x_5 1's followed by x_6 0's, where, $x_1+x_2+x_3+x_4+x_5+x_6=n$, $x_1, x_6 \geq 0$, $x_2, x_3, x_4, x_5 > 0$.

The number of solutions to this equation equals the number of solutions to $y_1+y_2+y_3+y_4+y_5+y_6=n-4$, where $y_i \geq 0$ for $1 \leq i \leq 6$. This number is $\binom{6+(n-4)-1}{n-4} = \binom{n+1}{5}$.

Ex 1.4: (28.b)

For $n \geq 6$, a string with this structure has x_1 1's followed by x_2 0's followed by x_3 1's … followed by x_8 0's, where $x_1+x_2+\ldots+x_8=n$, $x_1, x_8 \geq 0$, $x_2, \ldots, x_7 > 0$

The number of solutions to this equation equals the number of solutions to $y_1+y_2+\ldots+y_8=n\text{-}6$, where $y_i \geq 0$ for $1 \leq i \leq 8$. This number is $\binom{8+(n-6)-1}{n-6} = \binom{n+1}{7}$.

Ex 1.4: $(28.c)_{1/2}$

(c) There are 2^n strings in total and $n+1$ strings where there are k 1's followed by $n-k$ 0's, for $k=0,1,2,\ldots,n$. These $n+1$ strings contain no occurrences of 01, so there are $2^n-(n+1)=2^n-\binom{n+1}{1}$ strings that contain at least one occurrence of 01. There are $\binom{n+1}{3}$ strings that contain (exactly) one occurrence of 01, $\binom{n+1}{5}$ strings with (exactly) two occurrences, $\binom{n+1}{7}$ strings with (exactly) three occurrences, ... , and for

(i) n odd, we can have at most $\frac{n-1}{2}$ occurrences of 01. The number of strings with $\frac{n-1}{2}$ occurrences of 01 is the number of integer solutions for

$$x_1+x_2+\cdots+x_{n+1}=n,\ x_1,x_{n+1}\geq 0,\quad x_2,x_3,\ldots,x_n>0.$$

This is the same as the number of integer solutions for

$$y_1+y_2+\cdots+y_{n+1}=n-(n-1)=1,\ \text{ where }\ y_1,y_2,\ldots,y_{n+1}\geq 0.$$

This number is $\binom{(n+1)+1-1}{1}=\binom{n+1}{1}=\binom{n+1}{n}=\binom{n+1}{2(\frac{n-1}{2})+1}$.

Ex 1.4: $(28.c)_{2/2}$

(ii) n even, we can have at most $\frac{n}{2}$ occurrences of 01. The number of strings with $\frac{n}{2}$ occurrences of 01 is the number of integer solutions for

$$x_1 + x_2 + \cdots + x_{n+2} = n, \quad x_1, x_{n+2} \geq 0, \quad x_2, x_3, \ldots, x_n > 0.$$

This is the same as the number of integer solutions for

$$y_1 + y_2 + \cdots + y_{n+2} = n - n = 0, \quad \text{where } y_i \geq 0 \text{ for } 1 \leq i \leq n+2.$$

This number is $\binom{(n+2)+0-1}{0} = \binom{n+1}{0} = \binom{n+1}{n+1} = \binom{n+1}{2(\frac{n}{2})+1}$.
Consequently,

$$2^n - \binom{n+1}{1} = \binom{n+1}{3} + \binom{n+1}{5} + \cdots + \begin{cases} \binom{n+1}{n}, & n \text{ odd} \\ \binom{n+1}{n+1}, & n \text{ even}, \end{cases}$$

and the result follows.