Department of Computer Science
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CS 2336: Discrete Mathematics

Chapter 3

Set Theory

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Outline

3.1 Sets and Subsets

3.2 Set Operations and the Laws of Set Theory

3.3 Counting and Venn Diagrams

3.4 A First Word on Probability

Set and Element

- Set: A well-defined collection of objects. We use upper-case letters to denote sets, such as *A*, *B*, ……
- Element (Member): The objects contained in sets. We use lower-case letter to denote elements, such as *a*, *b*, ...

- We write $a \in A$ if *a* is an element of *A*, and $a \notin A$ if *a* is not an element of *A*

Example 3.1

- One way to represent a set is to use set braces
- Let A be a set of the five smallest positive integer
  - We write $A = \{1, 2, 3, 4, 5\}$
  - 1 is in A: $1 \in A$
  - 8 is not in A: $8 \notin A$
- Another way to represent A
  - $A = \{x | 1 \leq x \leq 5, x \in \mathbb{Z}\}$
  - It reads: the set of all x such that …
  - When the universe is clear (to be integers), we may write $A = \{x | 1 \leq x \leq 5\}$

Cardinality

- Sets can be finite or infinite set
  - $\{x | x > 0, x \in \mathbb{Z}\}$
  - $\{x | 1 > x > 0, x \in \mathbb{R}\}$

- For a finite set A, we use $|A|$ to denote the number of elements in it. It is called cardinality or size

Definition 3.1

- For two sets C and D from the same universe, C is a subset of D if and only if every element of C is an element of D
  - We write $C \subseteq D$ or $D \supseteq C$

- In addition, if D contains at least one element that is not in C, we call C is a proper subset of D
  - We write $C \subset D$ or $D \supset C$

Some Properties

- $C \subseteq D$ iff $\forall x [x \in C \Rightarrow x \in D]$
- For all C and D, $C \subset D \Rightarrow C \subseteq D$ and $D \supset C \Rightarrow D \supseteq C$
- For all C and D, $C \subseteq D \Rightarrow |C| \leq |D|$ and $C \subset D \Rightarrow |C| < |D|$

Definition 3.2

- For any sets A and B from the same universe, A and B are equal iff $A \subseteq B$ and $A \subseteq B$, we write $A = B$
  - Example: $\{1, 2, 3\} = \{3, 2, 1\} = \{2, 2, 1, 3\} = \{1, 2, 3, 1, 1\}$

Theorem 3.1

- Let A, B, and C be from the same universe

- If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$
- If $A \subset B$ and $B \subset C$, then $A \subset C$
- If $A \subseteq B$ and $B \subset C$, then $A \subset C$
- If $A \subset B$ and $B \subseteq C$, then $A \subset C$

Definition 3.3

- The null set, or empty set, is the (unique) set containing no elements.
- We denote it as $\{\}$ or $\emptyset$
- $|\emptyset| = 0$
- $\emptyset \neq \{0\}$
- $\emptyset \neq \{\emptyset\}$

Theorem 3.2

- For any universe $\mathbb{U}$, for $A \subseteq \mathbb{U}$, we have $\emptyset \subseteq A$
- Proof: Assume $\emptyset \not\subseteq A$, then there is an element x with $x \in \emptyset$ and $x \notin A$. However, $x \in \emptyset$ is impossible. Hence the assumption is rejected.
- Moreover, if $A \neq \emptyset$ then $\emptyset \subset A$

Example 3.7

- How many subsets does the set $C=\{1,2,3,4,5\}$ have?
- Approach #1: For each element, it can appear or not in a subset. Hence, C has $2^5 = 32$ subsets
- Approach #2: We may have *0*, *1*, *2*, …, *5* elements in a subset. $\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5} = 32$
- Definition 3.4: The power set of *A*, *P(A)*, is the collection of all subsets of *A*

Definition 3.5 & 3.6

- For A, B from the same universe, we define
  - Union: $A \cup B = \{x | x \in A \vee x \in B\}$
  - Intersection: $A \cap B = \{x | x \in A \wedge x \in B\}$
  - Symmetric Difference: $A \triangle B = \{x | x \in A \cup B \wedge x \notin A \cap B\}$
- Let S, T from the same universe. S and T are disjoint or mutually disjoint iff $S \cap T = \emptyset$

Definition 3.7 & 3.8

- For a set A from universe U, the complement of A, denoted by U-A or $\bar{A}$, which is given by
  $\{x|x \in U \wedge x \notin A\}$

- For set A and B from U, the (relative) complement of A in B, written as B-A, is given by
  $\{x|x \in B \wedge x \notin A\}$

- Let U be real numbers, $A = [1,2]$ and $B=[1,3)$. What are: (i) $A \cup B$, (ii) $A \cap B$, (iii) $\bar{A}$, and (iv) $B - A$

The Laws of Set Theory

For any sets A, B, and C taken from a universe $\mathcal{U}$

1) $\overline{\overline{A}} = A$	Law of *Double Complement*
2) $\overline{A \cup B} = \overline{A} \cap \overline{B}$	*DeMorgan's* Laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$	
3) $A \cup B = B \cup A$	*Commutative* Laws
$A \cap B = B \cap A$	
4) $A \cup (B \cup C) = (A \cup B) \cup C$	*Associative* Laws
$A \cap (B \cap C) = (A \cap B) \cap C$	
5) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	*Distributive* Laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	
6) $A \cup A = A$	*Idempotent* Laws
$A \cap A = A$	
7) $A \cup \emptyset = A$	*Identity* Laws
$A \cap \mathcal{U} = A$	
8) $A \cup \overline{A} = \mathcal{U}$	*Inverse* Laws
$A \cap \overline{A} = \emptyset$	
9) $A \cup \mathcal{U} = \mathcal{U}$	*Domination* Laws
$A \cap \emptyset = \emptyset$	
10) $A \cup (A \cap B) = A$	*Absorption* Laws
$A \cap (A \cup B) = A$	

Definition 3.9 and Theorem 3.5

- Let s be an equality statement of two set expression with only union and interactions operands. The dual of s, written as s^d can be derived from s by replacing: (i) each $\emptyset$ and U by U and $\emptyset$; (ii) each $\cup$ and $\cap$ by $\cap$ and $\cup$

- The principle of duality: let s be a theorem with the quality of two set expressions, then s^d is also a theorem

Definition 3.10

- Let I be a nonempty set and U be a universe. For each i in I, let $A_i \subseteq U$. Then I is called an index set, and each $i \in I$ is an index. Define
  - $\cup_{i \in I} A_i = \{x | x \in A_i \text{ for at least an } i \in I\}$
  - $\cap_{i \in I} A_i = \{x | x \in A_i \text{ for all } i \in I\}$

- Example: Let $U = \mathbb{R}$ and $I = \mathbb{R}^+,\ A_r = [-r, r]$, what are: (i) $\cup_{r \in I} A_r$ and (ii) $\cap_{r \in I} A_r$

Venn Diagrams

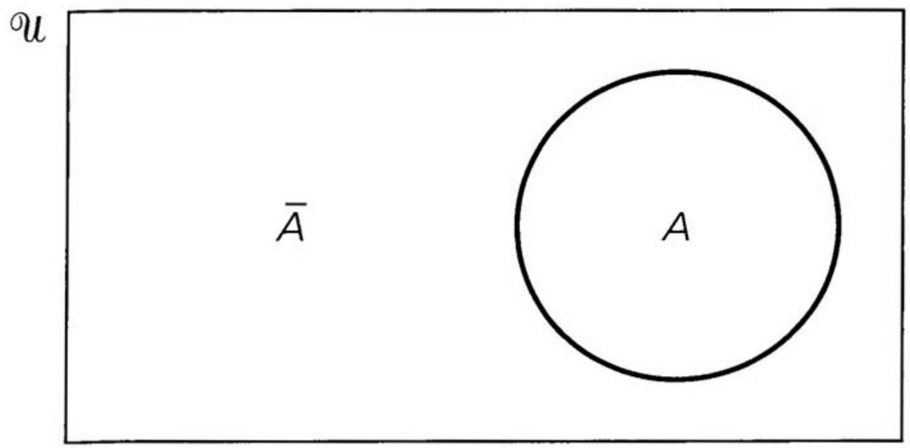

Figure 3.9

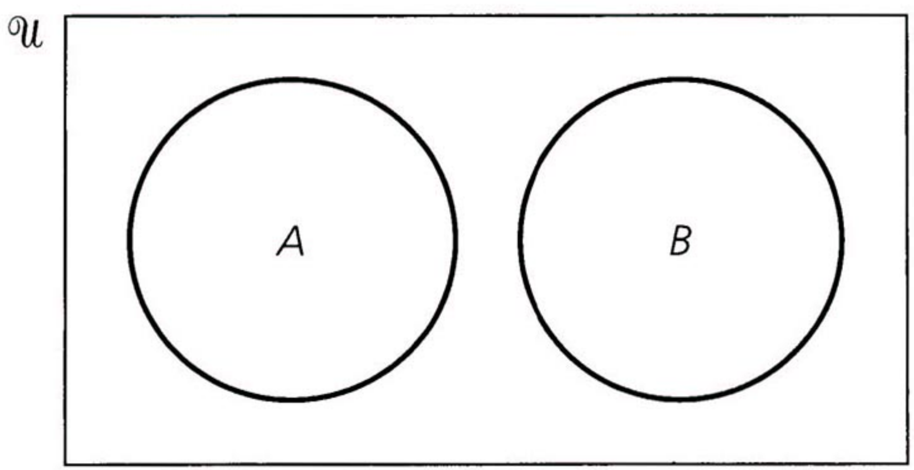

Figure 3.10

Counting

- For two finite sets: $|A \cup B| = |A| + |B| - |A \cap B|$
- If A and B are disjoint: $|A \cup B| = |A| + |B|$
-

$$|A \cup B \cup C| =$$

$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

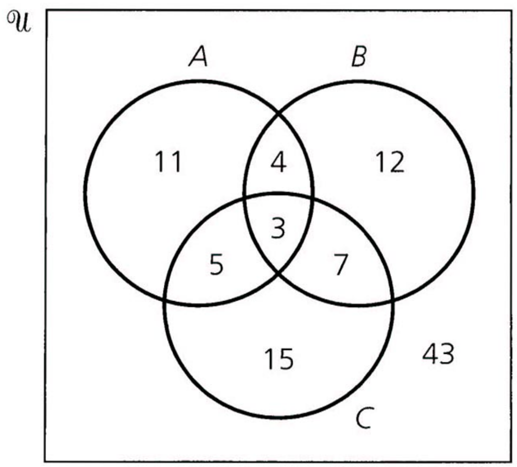

Figure 3.13

A First Word on Probability

- Example Experiments: toss a fair coin, roll a fair die, or randomly select 2 students from a class of 20
- Outcome: The item that got picked
- Sample Spaces ($\mathscr{S}$): the sets of all possible outcomes: {H, T}, {1, 2, 3, 4, 5, 6}, and {(i, j)| 1 <= i, j <=20}

Probability

- Assume equal likelihood, let $\mathscr{S}$ be the sample space for an experiment $\mathscr{E}$. Each subset A of $\mathscr{S}$ is called an event. Each element of $\mathscr{S}$ determines an outcome. Let $|\mathscr{S}| = n, A \subseteq \mathscr{S}$, $a \in \mathscr{S}$

  - Pr({a}) = The probability that {a} occurs $= \dfrac{|\{a\}|}{|\mathscr{S}|} = \dfrac{1}{n}$

  - Pr(A) = The probability that A occurs $= \dfrac{|A|}{|\mathscr{S}|} = \dfrac{|A|}{n}$

Cartesian Product

- For sets A, B, their Cartesian product, or cross product, is written as $A \times B = \{(a, b) | a \in A, b \in B\}$
- Consider an experiment: A single die is rolled and a coin is flipped. Both outcomes are noted.
  - Independent assumption

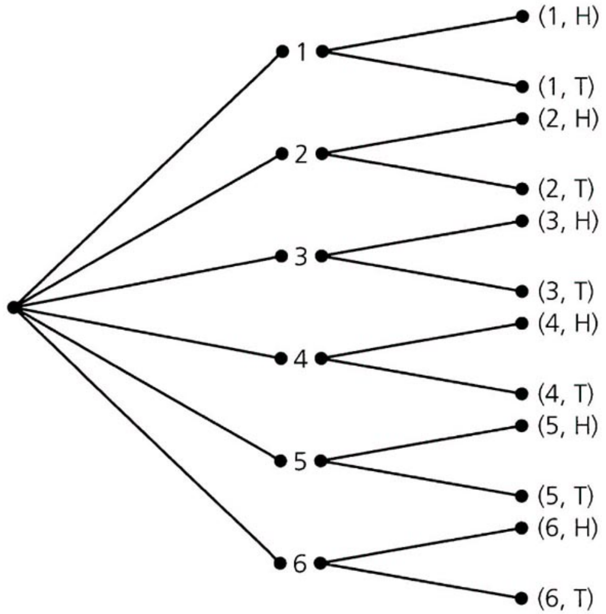

Figure 3.14

Take-home Exercises

- Exercise 3.1: 2, 5, 10, 15, 29
- Exercise 3.2: 2, 4, 7, 17, 19
- Exercise 3.3: 4, 5, 6, 10
- Exercise 3.4: 4, 8, 9, 11, 15