

SOLUTION

Ex 9.1: 1, 3, 4

Ex 9.2: 1, 2, 9, 18, 19

Ex 9.3: 1, 3, 4, 6

Ex 9.4: 1, 2, 5, 6, 9

Ex 9.5: 1, 2, 3, 5

Ex 9.1: (1)

- The number of integer solutions for the given equations is the coefficient of
 - a) x^{20} in $(1 + x + x^2 + \dots + x^7)^4$.
 - b) x^{20} in $(1 + x + x^2 + \dots + x^{20})^2(1 + x^2 + x^4 + \dots + x^{20})^2$
or $(1 + x + x^2 + \dots)^2(1 + x^2 + x^4 + \dots)^2$.
 - c) x^{30} in $(x^2 + x^3 + x^4)(x^3 + x^4 + \dots + x^8)^4$.
 - d) x^{30} in $(1 + x + x^2 + \dots + x^{30})^3$
 $(1 + x^2 + x^4 + \dots + x^{30})(x + x^3 + x^5 + \dots + x^{29})$ or
 $(1 + x + x^2 + \dots)^3(1 + x^2 + x^4 + \dots)(x + x^3 + x^5 + \dots)$.

Ex 9.1: (3)

- a) The generating function is either $(1 + x + x^2 + \cdots + x^{10})^6$ or $(1 + x + x^2 + \cdots)^6$.
[The number of ways to select 10 candy bars is the coefficient of x^{10} in either case.]
- b) The generating function is either $(1 + x + x^2 + \cdots + x^r)^n$ or $(1 + x + x^2 + \cdots)^n$.
[The number of selections of r objects is the coefficient of x^r in either case.]

Ex 9.1: (4)

- a) The first factor counts the pennies; the nickels are counted by the second factor.
- b) $f(x) = (1 + x + x^2 + \dots)(1 + x^5 + x^{10} + \dots)$
 $(1 + x^{10} + x^{20} + \dots)$.

Ex 9.2: (1)

a) $(1 + x)^8$

b) $8(1 + x)^7$

c) $(1 + x)^{-1}$

d) $\frac{6x^3}{1+x}$

e) $(1 - x^2)^{-1}$

f) $\frac{x^2}{1-ax}$

Ex 9.2: (2)

a) $-27, 54, -36, 8, 0, 0, 0, \dots$

b) $0, 0, 0, 0, 1, 1, 1, 1, 1, \dots$

c) $f(x) = \frac{x^3}{1-x^2} = x^3[1 + x^2 + x^4 + x^6 \dots] = x^3 + x^5 + x^7 + x^9 \dots$,
so $f(x)$ generates the sequence $0, 0, 0, 1, 0, 1, 0, 1, 0, 1, \dots$

d) $f(x) = \frac{1}{1+3x} = 1 + (-3x) + (-3x)^2 + (-3x)^3 + \dots$, so $f(x)$
generates the sequence $1, -3, 3^2, (-3)^3, \dots$

e) $f(x) = \frac{1}{3-x} = \left(\frac{1}{3}\right) \left[\frac{1}{1-\frac{x}{3}}\right] = \left(\frac{1}{3}\right) \left[1 + \frac{x}{3} + \left(\frac{x}{3}\right)^2 + \left(\frac{x}{3}\right)^3 \dots\right]$, so $f(x)$
generates the sequence $\frac{1}{3}, \left(\frac{1}{3}\right)^2, \left(\frac{1}{3}\right)^3, \left(\frac{1}{3}\right)^4, \dots$

f) $f(x) = \frac{1}{1-x} + 3x^7 - 11 = (1 + x + x^2 + x^3 + \dots) + 3x^7 - 11$, so
 $f(x)$ generates the sequence a_0, a_1, a_2, \dots , where $a_0 = -10$,
 $a_7 = 4$, and $a_i = 1$ for all $i \neq 0, 7$

Ex 9.2: (9)

a) 0

b) $\binom{-3}{12}(-1)^{12} - 5\binom{-3}{14}(-1)^{14} = \binom{14}{12} - 5\binom{16}{14}$

c) $\binom{-4}{15}(-1)^{15} + \binom{4}{1}\binom{-4}{14}(-1)^{14} + \binom{4}{2}\binom{-4}{13}(-1)^{13} +$
 $\binom{4}{3}\binom{-4}{12}(-1)^{12} + \binom{4}{4}\binom{-4}{11}(-1)^{11} = \binom{18}{15} + \binom{4}{1}\binom{17}{14} +$
 $\binom{4}{2}\binom{16}{13} + \binom{4}{3}\binom{15}{12} + \binom{14}{11}$

Ex 9.2: (18)

- $(1 - 4x)^{-\frac{1}{2}} = \left[\binom{-\frac{1}{2}}{0} + \binom{-\frac{1}{2}}{1} (-4x) + \binom{-\frac{1}{2}}{2} (-4x)^2 + \dots \right]$. The coefficient of x^n is

$$\begin{aligned} \binom{-\frac{1}{2}}{n} (-4)^n &= \frac{\left[\left((-\frac{1}{2}) - n + 1 \right) \left((-\frac{1}{2}) - n + 2 \right) \dots \left((-\frac{1}{2}) - 1 \right) \left(-\frac{1}{2} \right) \right]}{n!} (-4)^n = \\ &= \frac{[(1+2n-2)(1+2n-4)\dots(1+2)(1)]}{n!} (2)^n = \\ &= \frac{[(2n-1)(2n-3)\dots(5)(3)(1)]}{n!} (2)^n = \frac{[(2n-1)(2n-3)\dots(5)(3)(1)](2^n)(n!)}{n!n!} = \\ &= \frac{(2n)!}{n!n!} = \binom{2n}{n} \end{aligned}$$

Ex 9.2: (19)

- a) There are $2^{8-1} = 2^7$ compositions of 8 and $2^{\lfloor \frac{8}{2} \rfloor} = 2^4$ palindromes of 8. Assuming each composition of 8 has the same probability of being generated, the probability a palindrome of 8 is generated is $\frac{2^4}{2^7} = \frac{1}{8}$.
- b) Assuming each composition of n has the same probability of being generated, the probability a palindrome of n is generated is $\frac{2^{\lfloor \frac{n}{2} \rfloor}}{2^{n-1}} = 2^{\lfloor \frac{n}{2} \rfloor - n + 1} = 2^{1 - \lceil \frac{n}{2} \rceil}$.

Ex 9.3: (1)

- 7;

$$6 + 1;$$

$$5 + 2;$$

$$5 + 1 + 1;$$

$$4 + 3;$$

$$4 + 2 + 1;$$

$$4 + 1 + 1 + 1;$$

$$3 + 3 + 1;$$

$$3 + 2 + 2;$$

$$3 + 2 + 1 + 1;$$

$$3 + 1 + 1 + 1 + 1;$$

$$2 + 2 + 2 + 1;$$

$$2 + 2 + 1 + 1 + 1;$$

$$2 + 1 + 1 + 1 + 1 + 1;$$

$$1 + 1 + 1 + 1 + 1 + 1 + 1$$

Ex 9.3: (3)

- The number of partitions of 6 into 1's, 2's, and 3's is 7.

Ex 9.3: (4)

$$\text{a) } \left[\frac{1}{1-t^2} \right] \left[\frac{1}{1-t^3} \right] \left[\frac{1}{1-t^5} \right] \left[\frac{1}{1-t^7} \right]$$

$$\text{b) } \left[\frac{1}{1-t^2} \right] \left[\frac{t^{12}}{1-t^3} \right] \left[\frac{t^{20}}{1-t^5} \right] \left[\frac{t^{35}}{1-t^7} \right]$$

Ex 9.3: (6)

a) $f(x) = (1 + x + x^2 + \cdots + x^5)(1 + x^2 + x^4 + \cdots + x^{10}) \dots$
 $= \prod_{i=1}^{\infty} (1 + x^i + x^{2i} + \cdots + x^{5i}) = \prod_{i=1}^{\infty} \frac{1-x^{6i}}{1-x^i}$

b) $\prod_{i=1}^{12} (1 + x^i + x^{2i} + \cdots + x^{5i}) = \prod_{i=1}^{12} \frac{1-x^{6i}}{1-x^i}$

Ex 9.4: (1)

a) e^{-x}

b) e^{2x}

c) e^{-ax}

d) e^{a^2x}

e) ae^{a^2x}

f) xe^{2x}

Ex 9.4: (2)

- a) $f(x) = 3e^{3x} = 3 \sum_{i=0}^{\infty} \frac{(3x)^i}{i!}$, so $f(x)$ is the exponential generating function for the sequence $3, 3^2, 3^3, \dots$
- b) $f(x) = 6e^{5x} - 3e^{2x} = 6 \sum_{i=0}^{\infty} \frac{(5x)^i}{i!} - 3 \sum_{i=0}^{\infty} \frac{(2x)^i}{i!}$, so $f(x)$ is the exponential generating function for the sequence $3, 24, 138, \dots, 6(5^n) - 3(2^n), \dots$
- c) $1, 1, 3, 1, 1, 1, 1, \dots$
- d) $1, 9, 14, -10, 2^4, 2^5, 2^6, \dots$
- e) $f(x) = 1 + x + x^2 + \dots = \sum_{i=0}^{\infty} i! \left(\frac{x}{i!}\right)$, so $f(x)$ is the exponential generating function for the sequence $0!, 1!, 2!, 3!, \dots$
- f) $f(x) = 3[1 + 2x + (2x)^2 + \dots] + \sum_{i=0}^{\infty} \frac{x^i}{i!}$, so $f(x)$ is the exponential generating function for the sequence $4, 7, 25, 145, \dots, (3n!)2^n + 1, \dots$

Ex 9.4: (5)

- We find that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 \dots = 0! \frac{x^0}{0!} + 1! \frac{x^1}{1!} + 2! \frac{x^2}{2!} + (3!) \frac{x^3}{3!} \dots,$$

so $\frac{1}{1-x}$ is the exponential generating function for the sequence
 $0!, 1!, 2!, 3! \dots$

Ex 9.4: (6)

a) (i) $(1+x)^2 \left(1+x+\left(\frac{x^2}{2!}\right)\right)^2$

(ii) $(1+x) \left(1+x+\left(\frac{x^2}{2!}\right)\right) \left(1+x+\left(\frac{x^2}{2!}\right)+\left(\frac{x^3}{3!}\right)+\left(\frac{x^4}{4!}\right)\right)^2$

(iii) $(1+x)^3 \left(1+x+\left(\frac{x^2}{2!}\right)\right)^4$

b) $(1+x) \left(1+x+\left(\frac{x^2}{2!}\right)\right) \left(1+x+\left(\frac{x^2}{2!}\right)+\left(\frac{x^3}{3!}\right)+\left(\frac{x^4}{4!}\right)\right) \left(\left(\frac{x^2}{2!}\right)+\left(\frac{x^3}{3!}\right)+\left(\frac{x^4}{4!}\right)\right)$

Ex 9.4: (9)

$$\text{a) } \frac{1}{2} \frac{[3^{20}+1]}{3^{20}}$$

$$\text{b) } \frac{1}{4} \frac{[3^{20}+3]}{3^{20}}$$

$$\text{c) } \frac{1}{2} \frac{[3^{20}-1]}{3^{20}}$$

$$\text{d) } \frac{1}{2} \frac{[3^{20}-1]}{3^{20}}$$

$$\text{e) } \frac{1}{2} \frac{[3^{20}+1]}{3^{20}}$$

Ex 9.5: (1)

- a) $1 + x + x^2$ is the generating function for the sequence $1, 1, 1, 0, 0, 0, \dots$, so $\frac{1+x+x^2}{1-x}$ is the generating function for the sequence $1, 1 + 1, 1 + 1 + 1, 1 + 1 + 1 + 0, \dots$ – that is the sequence $1, 2, 3, 3, \dots$.
- b) $1 + x + x^2 + x^3$ is the generating function for the sequence $1, 1, 1, 1, 0, 0, 0, \dots$, so $\frac{1+x+x^2+x^3}{1-x}$ is the generating function for the sequence $1, 1 + 1, 1 + 1 + 1, 1 + 1 + 1 + 1, 1 + 1 + 1 + 1 + 0, 1 + 1 + 1 + 1 + 0 + 0, \dots$ – that is, the sequence $1, 2, 3, 4, 4, 4, \dots$.
- c) $1 + 2x$ is the generating function for the sequence $1, 2, 0, 0, 0, 0, \dots$, so $\frac{1+2x}{1-x}$ is the generating function for the sequence $1, 1 + 2, 1 + 2 + 0, 1 + 2 + 0 + 0, \dots$ – that is, the sequence $1, 3, 3, 3, \dots$. Consequently, $\frac{1}{1-x} \left[\frac{1+2x}{1-x} \right] = \frac{1+2x}{(1-x)^2}$ is the generating function for the sequence $1, 1 + 3, 1 + 3 + 3, 1 + 3 + 3 + 3 + 3, \dots$ – that is, the sequence $1, 4, 7, 10, \dots$.

Ex 9.5: (2)

a) (i) x

(ii) $\frac{x}{1-x}$

(iii) $\frac{x}{(1-x)^2}$

(iv) $\frac{x}{(1-x)^3}$

b) $\sum_{k=1}^n k =$ the coefficient of x^n in $\frac{x}{(1-x)^3}$
 $=$ the coefficient of x^n in $x(1-x)^{-3}$
 $=$ the coefficient of x^{n-1} in $(1-x)^{-3}$
 $= \binom{-3}{n-1} (-1)^{n-1} = (-1)^{n-1} \binom{3+(n-1)-1}{n-1} (-1)^{n-1}$
 $= \binom{n+1}{n-1} = \frac{1}{2}(n+1)n$

Ex 9.5: (3)

- $f(x) = \frac{[x(1+x)]}{(1-x)^3}$ generates $0^2, 1^2, 2^2, 3^2 \dots$;
 $\frac{[x(1+x)]}{(1-x)^3} = 0^2 + 1^2x + 2^2x^2 + 3^2x^3 + \dots$;
 $\left(\frac{d}{dx}\right) \left[\frac{x+x^2}{(1-x)^3} \right] = 1^3 + 2^3x + 3^3x^2 + \dots$;
 $x \left(\frac{d}{dx}\right) \left[\frac{x+x^2}{(1-x)^3} \right] = 0^3 + 1^3x + 2^3x^2 + 3^3x^3 + \dots$;
 $\left(\frac{d}{dx}\right) \left[\frac{x+x^2}{(1-x)^3} \right] = \frac{x^2+4x+1}{(1-x)^4}$, so $\frac{x(x^2+4x+1)}{(1-x)^5}$ generates $0^3, 0^3 + 1^3, 0^3 + 1^3 + 2^3, \dots$, and the coefficient of x^n is $\sum_{i=0}^n i^3$.
- $(x^3 + 4x^2 + x)(1-x)^{-5} = (x^3 + 4x^2 + x) \left[\binom{-5}{0} + \binom{-5}{1}(-x) + \binom{-5}{2}(-x)^2 + \dots \right]$.
 Here the coefficient of x^n is $\binom{-5}{n-3}(-1)^{n-3} + 4\binom{-5}{n-2}(-1)^{n-2} + \binom{-5}{n-1}(-1)^{n-1}$
 $= \binom{n+1}{n-3} + 4\binom{n+2}{n-2} + \binom{n+3}{n-1}$
 $= \left(\frac{1}{4!}\right) \left[\begin{array}{c} (n+1)n(n-1)(n-2) + \\ 4(n+2)(n+1)n(n-1) + (n+3)(n+2)(n+1)n \end{array} \right]$
 $= [(n+1)(n)/4!](6n^2 + 6n) = (1/4)(n+1)(n)(n^2 + n) = [(n+1)(n)/2]^2$

Ex 9.5: (5)

- $(1 - x)f(x) = (1 - x)(a_0 + a_1x + a_2x^2 + a_3x^3 \dots) = a_0 + (a_1 - a_0)x + (a_2 - a_1)x^2 + (a_3 - a_2)x^3 \dots$, so $(1 - x)f(x)$ is the generating function for the sequence $a_0, a_1 - a_0, a_2 - a_1, a_3 - a_2, \dots$.