# SOLUTION

Ex 8.1: 1, 6, 8, 16, 20

Ex 8.2: 2, 3, 8

Ex 8.3: 1, 4, 6, 9, 10

Ex 8.4 and Ex 8.5: 4, 5, 7, 8, 12

# Ex 8.1: (1)

- Let $x \in S$ and let $n$ be the number of conditions (from among $c_1, c_2, c_3, c_4$) satisfied by $x$. ($n = 0$): Here x is counted once in $N(\overline{c_2}\,\overline{c_3}\,\overline{c_4})$ and once in $N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4})$. ($n = 1$): If $x$ satisfies $c_1$ (and not $c_2, c_3, c_4$), then $x$ is counted once in $N(\overline{c_2}\,\overline{c_3}\,\overline{c_4})$ and once in $N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4})$
- If $x$ satisfies $c_i$, for $i \neq 1$, then $x$ is not counted in any of the three terms in the equation. ($n = 2,3,4$): If $x$ satisfies at least two of the four conditions, then $x$ is not counted in any of the three terms in the equation.
- The preceding observations show that the two sides of the given equation count the same elements from $S$, and this provides a combinational proof for the formula $N(\overline{c_2}\,\overline{c_3}\,\overline{c_4}) = N(c_1\overline{c_2}\,\overline{c_3}\,\overline{c_4}) + N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4})$

# Ex 8.1: (6a, 6b)

- $x_1 + x_2 + x_3 + x_4 = 19$

a) $0 \leq x_i, 1 \leq \text{i} \leq 4. \binom{4+19-1}{19} = \binom{22}{19}$

b) For $1 \leq \text{i} \leq 4$, let $c_i: x_i \geq 8$.
$N(c_i)$: $x_1 + x_2 + x_3 + x_4 = 11: \binom{4+11-1}{11} = \binom{14}{11}, 1 \leq i \leq 4.$
$N(c_i c_j)$: $x_1 + x_2 + x_3 + x_4 = 3: \binom{4+3-1}{3} = \binom{14}{3}, 1 \leq i < j \leq 4.$
$N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4}) = N - S_1 + S_2 = \binom{22}{19} - 4\binom{14}{11} + 6\binom{6}{3}.$

# Ex 8.1: (6c)

- The number of solutions for $x_1 + x_2 + x_3 + x_4 = 19$ where $0 \le x_1 \le 5, 0 \le x_2 \le 6, 3 \le x_3 \le 7, 3 \le x_4 \le 8$ equals the number of solutions for $x_1 + x_2 + x_3 + x_4 = 13$ with $0 \le x_1 \le 5, 0 \le x_2 \le 6, 0 \le x_3 \le 4, 0 \le x_4 \le 5$. Define the conditions $c_i$, $1 \le i \le 4$, as follows: $c_1: x_1 \ge 6, c_2: x_2 \ge 7; c_3: x_3 \ge 5; c_4: x_4 \ge 6$.
  $N = \binom{4+13-1}{13} = \binom{16}{13}$.
  $N(c_1), N(c_4): x_1 + x_2 + x_3 + x_4 = 7: \binom{4+7-1}{7} = \binom{10}{7}$.
  $N(c_2): x_1 + x_2 + x_3 + x_4 = 6: \binom{4+6-1}{6} = \binom{9}{6}$.
  $N(c_3): x_1 + x_2 + x_3 + x_4 = 8: \binom{4+8-1}{8} = \binom{11}{8}$.
  $N(c_1 c_2) = 1$.
  $N(c_1 c_3): x_1 + x_2 + x_3 + x_4 = 2: \binom{4+2-1}{2} = \binom{5}{2}$.
  $N(c_1 c_4): x_1 + x_2 + x_3 + x_4 = 1: \binom{4+1-1}{1} = \binom{4}{1}$.
  $N(c_2 c_3) = \binom{4}{1}, N(c_2 c_4) = 1, N(c_3 c_4) = \binom{5}{2}$.5
  $N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4}) = \binom{16}{13} - [2\binom{10}{7} + \binom{9}{6} + \binom{11}{8}] + 2[1 + \binom{4}{1} + \binom{5}{2}]$.

# Ex 8.1: (8)

- The number of integer solutions for $x_1 + x_2 + x_3 + x_4 = 19$, -$5 \leq \mathrm{x}_1 \leq 10, 1 \leq \mathrm{i} \leq 4$, equals the number of integer solutions for $y_1 + y_2 + y_3 + y_4 = 39$, $0 \leq \mathrm{y}_i \leq 15$.
- For $1 \leq \mathrm{i} \leq 4$, let $c_i$: $y_i \geq 16$.
  $N(c_i), 1 \leq i \leq 4$: $y_1 + y_2 + y_3 + y_4 = 23$: $\binom{4+23-1}{23} = \binom{26}{23}$.
  $N\left(c_i c_j\right), 1 \leq i < j \leq 4$: $y_1 + y_2 + y_3 + y_4 = 7$: $\binom{4+7-1}{7} = \binom{10}{7}$.
  $N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4}) = \binom{42}{39} - 4\binom{26}{23} + 6\binom{10}{7}$.

# Ex 8.1: (16)

- $10^9 - \binom{3}{1}9^9 + \binom{3}{2}8^9 - \binom{3}{3}7^9.$

# Ex 8.1: (20)

- For $1 \leq \mathrm{i} \leq 7$, let $c_i$ denote the situation where the $i$-th friend was at lunch with Sharon.
  Then $N(\overline{c_1}\,\overline{c_2} \ldots \overline{c_7}) =$
  $84 - \binom{7}{1}35 + \binom{7}{2}16 - \binom{7}{3}8 + \binom{7}{4}4 - \binom{7}{5}2 + \binom{7}{6}1 - \binom{7}{7}0 = 0.$
  Consequently, Sharon always had company at lunch.

# Ex 8.2: (2)

a) Let $c_i$ denote the condition that the two A's are together in an arrangement of ARRANGEMENT. Conditions $c_2, c_3, c_4$ are defined similarly for the two E's, N's, and R's, respectively.
$N = {}^{(11!)}/_{[(2!)^4]} = 2494800.$
For $1 \le \mathrm{i} \le 4, N(c_i) = \frac{10!}{(2!)^3} = 453600.$
For $1 \le \mathrm{i} < j \le 4, N(c_i c_j) = \frac{9!}{(2!)^2} = 90720.$
$N\left(c_i c_j c_k\right) = \frac{8!}{2!} = 20160, 1 \le \mathrm{i} < j < k \le 4.$
$N(c_1 c_2 c_3 c_4) = 7! = 5040.$
$S_1 = \binom{4}{1}453600 = 1814400. S_2 = \binom{4}{2}90720 = 544320.$
$S_3 = \binom{4}{3}20160 = 80640. S_4 = \binom{4}{4}5040 = 5040.$
(i) $E_2 = S_2 - \binom{3}{1}S_3 + \binom{4}{2}S_4 = 332640$
(ii) $L_2 = S_2 - \binom{2}{1}S_3 + \binom{3}{1}S_4 = 398160$

b) (i) $E_3 = S_3 - \binom{4}{1}S_4 = 60480.$ (ii) $L_3 = S_3 - \binom{3}{2}S_4 = 65520.$

# Ex 8.2: (3)

- Let $c_1$ denote the presence of consecutive E's in the arrangement. Likewise, $c_2$, $c_3, c_4$, and $c_5$ are defined for consecutive N's, O's, R's, and S's, respectively.

a) $N = \frac{14!}{(2!)^5}$

$N(c_1) = \frac{13!}{(2!)^4}$; $S_1 = \binom{5}{1}\left(\frac{13!}{(2!)^4}\right)$.

$N(c_1c_2) = \frac{12!}{(2!)^3}$; $S_2 = \binom{5}{2}\left(\frac{12!}{(2!)^3}\right)$.

$N(c_1c_2c_3) = \frac{11!}{(2!)^2}$; $S_3 = \binom{5}{3}\left(\frac{11!}{(2!)^2}\right)$.

$N(c_1c_2c_3c_4) = \frac{10!}{(2!)^1}$; $S_4 = \binom{5}{4}\left(\frac{10!}{(2!)^1}\right)$.

$N(c_1c_2c_3c_4c_5) = 9! = S_5$.

$N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4}\,\overline{c_5}) =1{,}286{,}046{,}720$

b) $E_2 = S_2 - \binom{3}{1}S_3 + \binom{4}{2}S_4 - \binom{5}{3}S_5 = 350{,}179{,}200$

c) $L_3 = S_3 - \binom{3}{2}S_4 + \binom{4}{2}S_5 = 74{,}753{,}280$

# Ex 8.2: (8)

b) $E_{t-1} = S_{t-1} - tS_t$; $L_{t-1} = L_t + E_{t-1}$

c) $L_{t-1} = L_t + E_{t-1} = S_t + S_{t-1} - tS_t = S_{t-1} - (t-1)S_t = S_{t-1} - \binom{t-1}{t-2}S_t$

d) $L_m = L_{m+1} + E_m$

e) $L_t = S_t$

$L_{t-1} = S_{t-1} - \binom{t-1}{t-2}S_t$

Assume $L_{k+1} = S_{k+1} - \binom{k+1}{k}S_{k+2} + \binom{k+2}{k}S_{k+3} - \cdots + (-1)^{t-k-1}\binom{t-1}{k}S_t$

$L_k = L_{k+1} + E_k =$
$[S_{k+1} - \binom{k+1}{k}S_{k+2} + \binom{k+2}{k}S_{k+3} - \cdots + (-1)^{t-k-1}\binom{t-1}{k}S_t]$
$+[S_k - \binom{k+1}{1}S_{k+1} + \binom{k+1}{2}S_{k+2} - \cdots + (-1)^{t-k}\binom{t}{t-k}S_t]$.

For $1 \leq \mathrm{r} \leq \mathrm{t} - k$, the coefficient of $S_{k+r}$ is $(-1)^{r-1}\binom{k+r-1}{k} + (-1)^r\binom{k+r}{r} = (-1)^r\binom{k+r-1}{k-1}$.

Consequently, $L_k = S_k - \binom{k}{k-1}S_{k+1} + \binom{k+1}{k-1}S_{k+2} - \cdots + (-1)^{t-k}\binom{t-1}{k-1}S_t$.

# Ex 8.3: (1)

- For $1 \leq \mathrm{i} \leq 5$ let $c_i$ be the condition that $2i$ is in position $2i$.
  $N = 10!$; $N(c_i) = 9!$; $N(c_i c_j) = 8!, 1 \leq \mathrm{i} < j \leq 5; \ldots;$
  $N(c_1 c_2 c_3 c_4 c_5) = 5!$.
  $N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4}\,\overline{c_5}) = 10! - \binom{5}{1}9! + \binom{5}{2}8! - \binom{5}{3}7! + \binom{5}{4}6! - \binom{5}{5}5!$

# Ex 8.3: (4)

- There are $7! = 5040$ permutations for $1,2,3,4,5,6,7$.
  Among these there are $7!\left[1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!}\right] = 1854$ derangements.
  Consequently, we have $5040 - 1854 = 3186$ permutations of $1,2,3,4,5,6,7$ that are not derangements.

# Ex 8.3: (6)

a) There are $(d_4)^2 = 9^2 = 81$ such derangements.
b) In this case we get $(4!)^2 = 24^2 = 576$ derangements.

# Ex 8.3: (9)

- $(10!)d_{10} \doteq (10!)^2(e^{-1})$

# Ex 8.3: (10)

a) (i) $\frac{d_n}{n!}$. (ii) $\frac{n(d_{n-1})}{n!}$. (iii) $1 - \frac{d_n}{n!}$. (iv) $\frac{\binom{n}{r}d_{n-r}}{n!}$.

b) (i) $e^{-1}$. (ii) $e^{-1}$. (iii) $1 - e^{-1}$. (iv) $\left(\frac{1}{r!}\right)e^{-1}$.

# Ex 8.4 and Ex 8.5: (4)

- $r(C_1, x) = 1 + 4x + 3x^2 = r(C_2, x)$

# Ex 8.4 and Ex 8.5: (5)

a) (i) $(1 + 2x)^3$
(ii) $1 + 8x + 12x^2 + 4x^3$
(iii) $1 + 9x + 25x^2 + 21x^3$
(iv) $1 + 8x + 16x^2 + 7x^3$

b) If the board $C$ consists of $n$ steps, and each step has $k$ blocks, then $r(C, x) = (1 + kx)^n$.

# Ex 8.4 and Ex 8.5: (7)

- $r(C, x) = (1 + 4x + 3x^2)(1 + 4x + 2x^2)$
  $= 1 + 8x + 21x^2 + 20x^3 + 6x^4$.
  For $1 \leq \mathrm{i} \leq 5$ let $c_i$ be the condition that an assignment is made with person $(i)$ assigned to a language he or she wishes to avoid.
  $N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4}\,\overline{c_5}) = 5! - 8 \times 4! + 21 \times 3! - 20 \times 2! + 6 \times 1! = 20.$

| | Java | C++ | VHDL | Perl | SQL |
|---|---|---|---|---|---|
| (1) Jeanne | | | | | |
| (2) Charles | | | | | |
| (3) Todd | | | | | |
| (4) Paul | | | | | |
| (5) Sandra | | | | | |

# Ex 8.4 and Ex 8.5: (8)

- The factor 6! is needed because we are counting ordered sequences.

# Ex 8.4 and Ex 8.5: (12)

- Consider the chessboard $C$ of shaded squares.
  Here $r(C, x) = 1 + 8x + 20x^2 + 17x^3 + 4x^4$. For any one-to-one function $f: A \to \mathrm{B}$, let $c_1, c_2, c_3, c_4$ denote the conditions:
  $c_1$: $f(1) = v$ or $w$ $\quad$ $c_3$: $f(3) = x$
  $c_2$: $f(2) = u$ or $w$ $\quad$ $c_4$: $f(4) = v, x,$ or $y$
  The answer to this problem is $N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4})$. So there are 146 one-to-one functions $f: A \to \mathrm{B}$ where
  $f(1) \neq \mathrm{v}, w$ $\quad$ $f(3) \neq \mathrm{x}$
  $f(2) \neq \mathrm{u}, w$ $\quad$ $f(4) \neq \mathrm{v}, x, y$.