# Solution
# Week 1

Ex 1.1 & 1.2: 15, 22, 28, 32, 33

Ex 1.3: 13, 16, 25, 29, 34

Ex 1.4: 7, 17, 24, 26, 28

# Ex 1.1 & 1.2: (15)

- Here we must place a, b, c, d in the positions denoted by $x$: e $\underline{x}$ e $\underline{x}$ e $\underline{x}$ e $\underline{x}$ e.
- By the rule of product, there are 4! ways to do this.

# Ex 1.1 & 1.2: (22)

* Case1: The leading digit is 5: (6!)/(2!)
* Case2: The leading digit is 6: $(6!)/(2!)^2$
* Case3: The leading digit is 7: $(6!)/(2!)^2$
* In total there are
  [(6!)/(2!)][1+(1/2)+(1/2)] = 720 such position integers n.

# Ex 1.1 & 1.2: (28)

a) The for loops for $i$, $j$, $k$ are executed 12, 6, 8 times, respectively. The value of counter is
$0 + 12\times1 + 6\times2 + 8\times3 = 48$.

b) By the rule of sum.

# Ex 1.1 & 1.2: (32)

a) For positive integers $n$, $k$ where $n = 3k$, then $n!/(3!)^k$ is the number of ways to arrange the following $n$ objects $x_1, x_1, x_1, x_2, x_2, x_2, \ldots, x_k, x_k, x_k$.
Therefore, it must be an integer.

b) If $n$, $k$ are positive integers with $n = mk$, then $n!/(m!)^k$ is an integer.

# Ex 1.1 & 1.2: (33)

a) With 2 choices per question.
   There are $2^{10} = 1024$ ways.

b) With 3 choices per question.
   There are $3^{10}$ ways.

# Ex 1.3: (13)

* The letters M,I,I,I,P,P,I can be arranged in [7!/(4!2!)] ways. Each arrangement provides 8 locations for placing the 4 S's in nonconsecutive ways.
* Four of S's locations from 8 possible locations can be selected in $\binom{8}{4}$ ways. Hence, total number of these arrangements is $\binom{8}{4}$[7!/(4!2!)].

# Ex 1.3: (16)

a) 97
b) -5
c) 12
d) 0
e) 3

# Ex 1.3: (25)

*a)* $\binom{4}{1,1,2} = 12$

*b)* $\binom{4}{0,1,1,2} = 12$

*c)* $\binom{4}{1,1,2}(2)(-1)(-1)^2 = -24$

*d)* $\binom{4}{1,1,2}(-2)(3)^2 = -216$

*e)* $\binom{8}{3,2,1,2}(2)^3(-1)^2(3)(-2)^2 = 161280$

# Ex 1.3: (29)

* $n\binom{m+n}{m}$

$$= n\frac{(m+n)!}{m!\,n!}$$

$$= \frac{(m+n)!}{m!\,(n-1)!}$$

$$= (m+1)\frac{(m+n)!}{(m+1)(m!)(n-1)!}$$

$$= (m+1)\binom{m+n}{m+1}$$

# Ex 1.3: (34)

a) **procedure** *Select2* (*i*,*j*: positive integers)

```
begin
    for i := 1 to 5 do
            for j := i + 1 to 6 do
                    print (i,j)
end
```

b) **procedure** *Select3* (*i*,*j*,*k*: positive integers)

```
begin
    for i := 1 to 4 do
            for j := i + 1 to 5 do
                    for k := j + 1 to 6 do
                            print (i,j,k)
end
```

# Ex 1.4: (7)

a) $\binom{4+32-1}{32} = \binom{35}{32}$

b) $\binom{4+28-1}{28} = \binom{31}{28}$

c) $\binom{4+8-1}{8} = \binom{11}{8}$

d) 1

e) Let $y_i = x_i + 2$, $1 \le i \le 4$. The number of solutions to the given problem is then the same as the number of solutions to $y_1+y_2+y_3+y_4 = 40$, $0 \le y_i$, $1 \le i \le 4$. $\binom{4+40-1}{40} = \binom{43}{40}$.

f) $\binom{4+28-1}{28} - \binom{4+3-1}{3} = \binom{31}{28} - \binom{6}{3}$, where the term $\binom{6}{3}$ accounts for the solutions where $26 \le x_4$.

# Ex 1.4: (17)

a) $\binom{5+12-1}{12} = \binom{16}{12}$

b) $5^{12}$

# Ex 1.4: (24)

a) **procedure** Selection1 ($i,j$: nonnegative integers)

```
begin
    for i := 0 to 10 do
        for j := 0 to 10 – i do
            print (i,j,10-i-j)
end
```

b) Let $y_i = x_i + 2 \geq 0$. It's equal to solve
$y_1+y_2+y_3+y_4 = 12$, where $y_i \geq 0$ for $1 \leq i \leq 4$.
The algorithm is like (a).

# Ex 1.4: (26)

* Each such composition can be factored as $k$ times a composition of $m$.
* Consequently, there are $2^{m-1}$ compositions of $n$, where $n = mk$ and each summand in a composition is a multiple of $k$.

# Ex 1.4: (28.a)

A string of this type consists of $x_1$ 1's followed by $x_2$ 0's followed by $x_3$ 1's followed by $x_4$ 0's followed by $x_5$ 1's followed by $x_6$ 0's, where, $x_1+x_2+x_3+x_4+x_5+x_6=n$, $x_1,x_6\geq 0$, $x_2,x_3,x_4,x_5>0$.

The number of solutions to this equation equals the number of solutions to $y_1+y_2+y_3+y_4+y_5+y_6=n-4$, where $y_i\geq 0$ for $1\leq i\leq 6$. This number is $\binom{6+(n-4)-1}{n-4}=\binom{n+1}{5}$.

# Ex 1.4: (28.b)

For n≥6, a string with this structure has $x_1$ 1's followed by $x_2$ 0's followed by $x_3$ 1's … followed by $x_8$ 0's, where $x_1+x_2+\ldots+x_8=n$, $x_1,x_8\geq 0$, $x_2,\ldots,x_7>0$

The number of solutions to this equation equals the number of solutions to $y_1+y_2+\ldots+y_8=n\text{-}6$, where $y_i\geq 0$ for $1\leq i\leq 8$. This number is $\binom{8+(n-6)-1}{n-6}=\binom{n+1}{7}$.

# Ex 1.4: $(28.c)_{1/2}$

(c) There are $2^n$ strings in total and $n+1$ strings where there are $k$ 1's followed by $n-k$ 0's, for $k = 0, 1, 2, \ldots, n$. These $n+1$ strings contain no occurrences of 01, so there are $2^n - (n+1) = 2^n - \binom{n+1}{1}$ strings that contain at least one occurrence of 01. There are $\binom{n+1}{3}$ strings that contain (exactly) one occurrence of 01, $\binom{n+1}{5}$ strings with (exactly) two occurrences, $\binom{n+1}{7}$ strings with (exactly) three occurrences, ... , and for
(i) $n$ odd, we can have at most $\frac{n-1}{2}$ occurrences of 01. The number of strings with $\frac{n-1}{2}$ occurrences of 01 is the number of integer solutions for

$$x_1 + x_2 + \cdots + x_{n+1} = n, \; x_1, x_{n+1} \geq 0, \quad x_2, x_3, \ldots, x_n > 0.$$

This is the same as the number of integer solutions for

$$y_1 + y_2 + \cdots + y_{n+1} = n - (n-1) = 1, \quad \text{where } \; y_1, y_2, \ldots, y_{n+1} \geq 0.$$

This number is $\binom{(n+1)+1-1}{1} = \binom{n+1}{1} = \binom{n+1}{n} = \binom{n+1}{2(\frac{n-1}{2})+1}$.

# Ex 1.4: $(28.c)_{2/2}$

(ii) $n$ even, we can have at most $\frac{n}{2}$ occurrences of 01. The number of strings with $\frac{n}{2}$ occurrences of 01 is the number of integer solutions for

$$x_1 + x_2 + \cdots + x_{n+2} = n, \quad x_1, x_{n+2} \geq 0, \quad x_2, x_3, \ldots, x_n > 0.$$

This is the same as the number of integer solutions for

$$y_1 + y_2 + \cdots + y_{n+2} = n - n = 0, \quad \text{where } y_i \geq 0 \text{ for } 1 \leq i \leq n+2.$$

This number is $\binom{(n+2)+0-1}{0} = \binom{n+1}{0} = \binom{n+1}{n+1} = \binom{n+1}{2(\frac{n}{2})+1}$.
Consequently,

$$2^n - \binom{n+1}{1} = \binom{n+1}{3} + \binom{n+1}{5} + \cdots + \begin{cases} \binom{n+1}{n}, & n \text{ odd} \\ \binom{n+1}{n+1}, & n \text{ even,} \end{cases}$$

and the result follows.