CS 2336: Discrete Mathematics

Chapter 11

An Introduction to Graph Theory

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Outline

11.1 Definitions and Examples

11.2 Subgraphs, Complements, and Graph Isomorphism

11.3 Vertex Degree: Euler Trails and Circuits

11.4 Planar Graphs

11.5 Hamilton Paths and Cycles

11.6 Graph Coloring and Chromatic Polynomials

Graph

- For a set of towns, which are connected by a set of roads, how can be find our way from our current location to the destination?
- We need a systematic way to describe the relation: town *A* can be reached from town *B* via road *R*
  - M1: Relations! But tedious.
  - M2: Map, which can be abstracted by a Graph

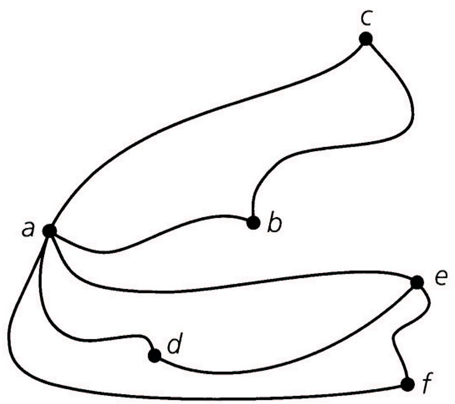

Figure 11.1

Definition

- Let V be the set of vertices, or nodes, and E is its (directed) edges, or arcs. $G=(V,E)$ is called a directed graph, or digraph.
- When edge direction is not important, we call G a undirected graph, and E now contains unordered pairs of vertices.
- V is called the vertex set and E is called the edge set.

Example on Directed Graph

- What are the vertex and edge sets in this graph?
- For (*b*,*c*), we say *c* is adjacent from *b*, and *b* is adjacent to *c*.
- For (*b*,*c*), b is the origin or source, and c is the terminus or terminating vertex
- An edge (*a*,*a*) is a loop and a vertex that has no incident edge is an isolated vertex

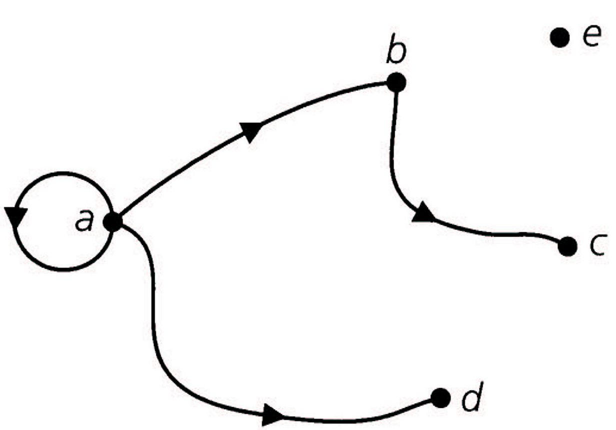

Figure 11.2

Example on Undirected Graph

- $\{a,b\}$ stands for $\{(a,b),(b,a)\}$

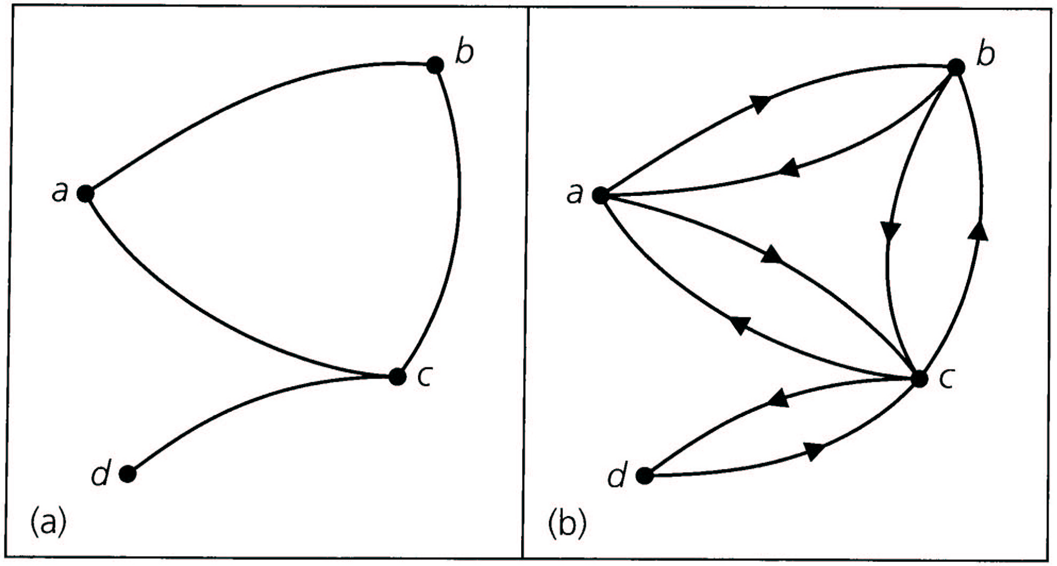

Figure 11.3

- If a graph is not specified as directed or undirected, we assume it's undirected
- Graphs with no loops are called loop-free

Walk

- Let x, y (not necessarily distinct) be vertices in a graph $G=(V,E)$. An x-y walk in G is a (loop-free) finite alternating sequence: $x=x_0,e_1,x_1,e_2,x_2,e_3,\ldots,e_{n-1},x_{n-1},e_n,x_n=y$ of vertices and edges.

- The length of this walk is n
  - If x=y, and the length is zero, the walk is called trivial.
  - If x=y, and length is nonzero, the walk is a closed one.

Example of Walks

- Ex 11.1: Three open walks
  - {a,b},{b,d},{d,c},{c,e} {e,d} {d,b}
  - b→c→d→e→c→f
  - {f,c},{c,e},{e,d},{d,a}
- Since the graph is undirected, an a-b walk is also a b-a walk!
- b→c→b is a b-b closed walk. Can we get a c-c closed walk?

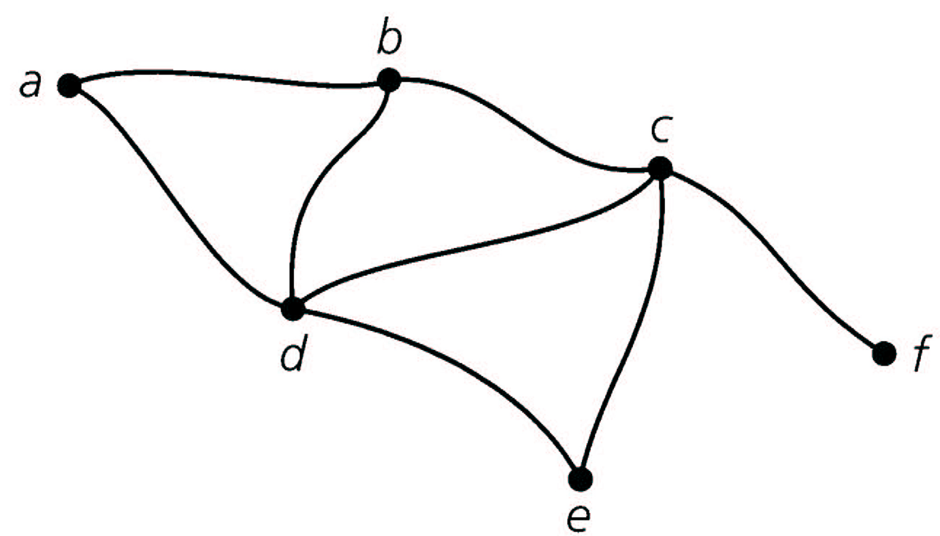

Figure 11.4

Trails and Paths

- For any x-y walk in G
  - If no edge in the x-y walk is repeated, we call it x-y trail
  - A closed trail is called a circuit
  - If no vertex of the x-y walk occurs more than once, we call it x-y path
  - A closed path is called a cycle

- Convention: Circuits have at least one edge. Cycles contain at least three edges.

Example of Trails and Paths

- Ex 11.2: Are the following two walks trails or paths?
  - b→c→d→e→c→f
  - {f,c},{c,e},{e,d},{d,a}
- Check if a→b→d→c→e→d→a is a circuit or cycle?
- a→b→c→d→a is an

  a-a cycle.

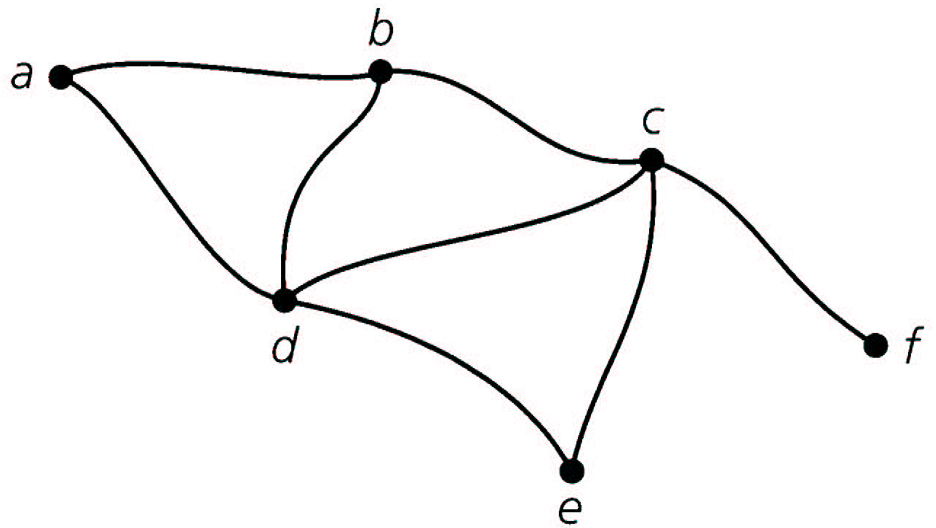

Figure 11.4

Summary

Table 11.1

Repeated Vertex (Vertices)	Repeated Edge(s)	Open	Closed	Name
Yes	Yes	Yes		Walk (open)
Yes	Yes		Yes	Walk (closed)
Yes	No	Yes		Trail
Yes	No		Yes	Circuit
No	No	Yes		Path
No	No		Yes	Cycle

Theorem

- Let $G=(V,E)$ be an undirected graph. For two distinct vertices a and b, if there is a trial from a to b, then there is a path from a to b.
- Proof Sketch:
  - We select the shortest trail from a to b: $a \rightarrow x_1 \rightarrow x_2 \rightarrow \ldots \rightarrow x_n \rightarrow b$
  - If it's a path, then we are done. Otherwise, we know the trail can be written as $a \rightarrow x_1 \rightarrow \ldots \rightarrow x_{k-1} \rightarrow x_k \rightarrow x_{k+1} \rightarrow \ldots \rightarrow k_{l-1} \rightarrow k_l \rightarrow k_{l+1} \rightarrow \ldots \rightarrow x_n \rightarrow b$, where $x_k=x_l$
  - Then we found a shorter trail from a to b, $a \rightarrow x_1 \rightarrow \ldots \rightarrow x_{k-1} \rightarrow x_k \rightarrow k_{l+1} \rightarrow \ldots \rightarrow x_n \rightarrow b$, contradiction!

Connected

- Let $G=(V,E)$ be an undirected graph. G is connected if there is a path between any two distinct vertices.
- For a directed graph G, it's associated undirected graph is the graph obtained from G by ignoring the directions of edges. G is connected if its associated undirected graph is connected.
- A graph that is not connected is called disconnected.

Connected Components

- Ex 11.3: Find the two connected components in the graph
  - A graph is connected if it has only one component
- The number of components of G is denoted by $\kappa(G)$

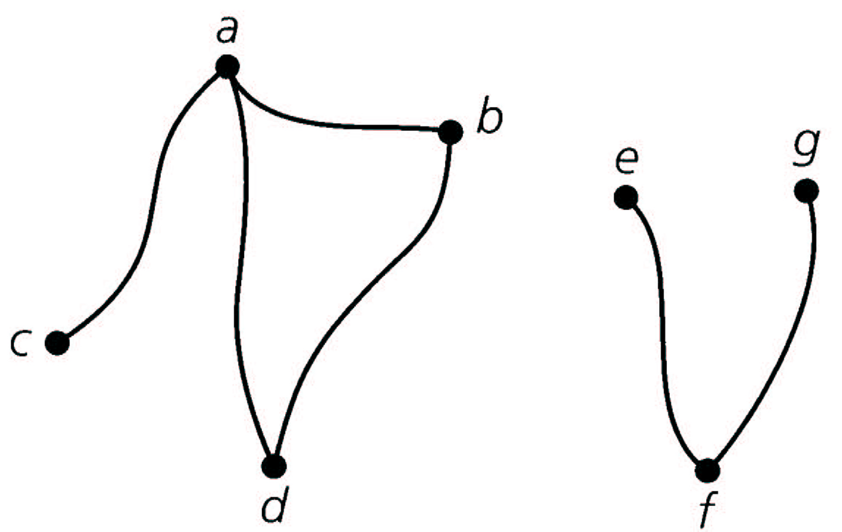

Figure 11.5

Multigraph

- (V,E) describes a multigraph G with vertex set V and edge set E if for some $x, y \in V$, there are two or more edges in E.
- Multigraphs can be directed or undirected.

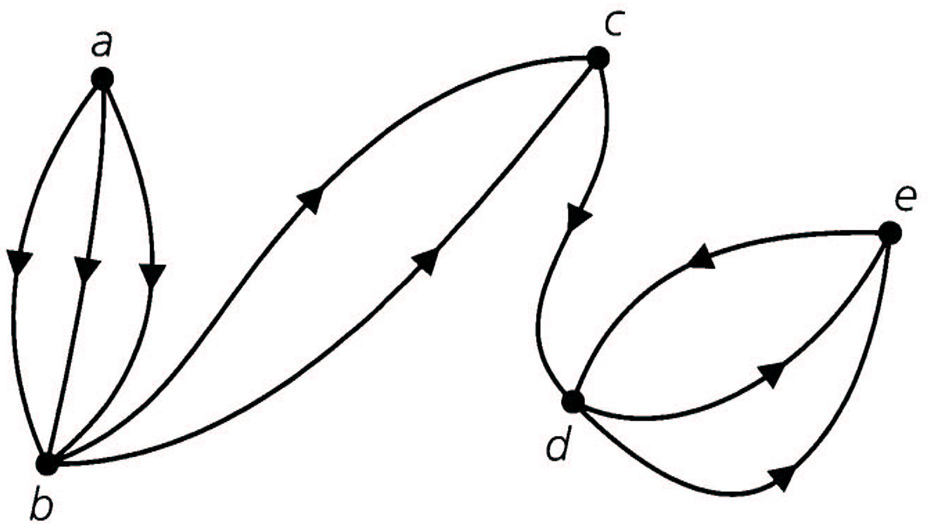

Figure 11.6

Outline

11.1 Definitions and Examples

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Subgraph

- If $G=(V,E)$ is a graph (directed or undirected), then $G_1=(V_1,E_1)$ is a subgraph of G if $\varnothing \neq V_1 \subseteq V, E_1 \subseteq E$, where each edge in E_1 is incident with vertices in V_1.

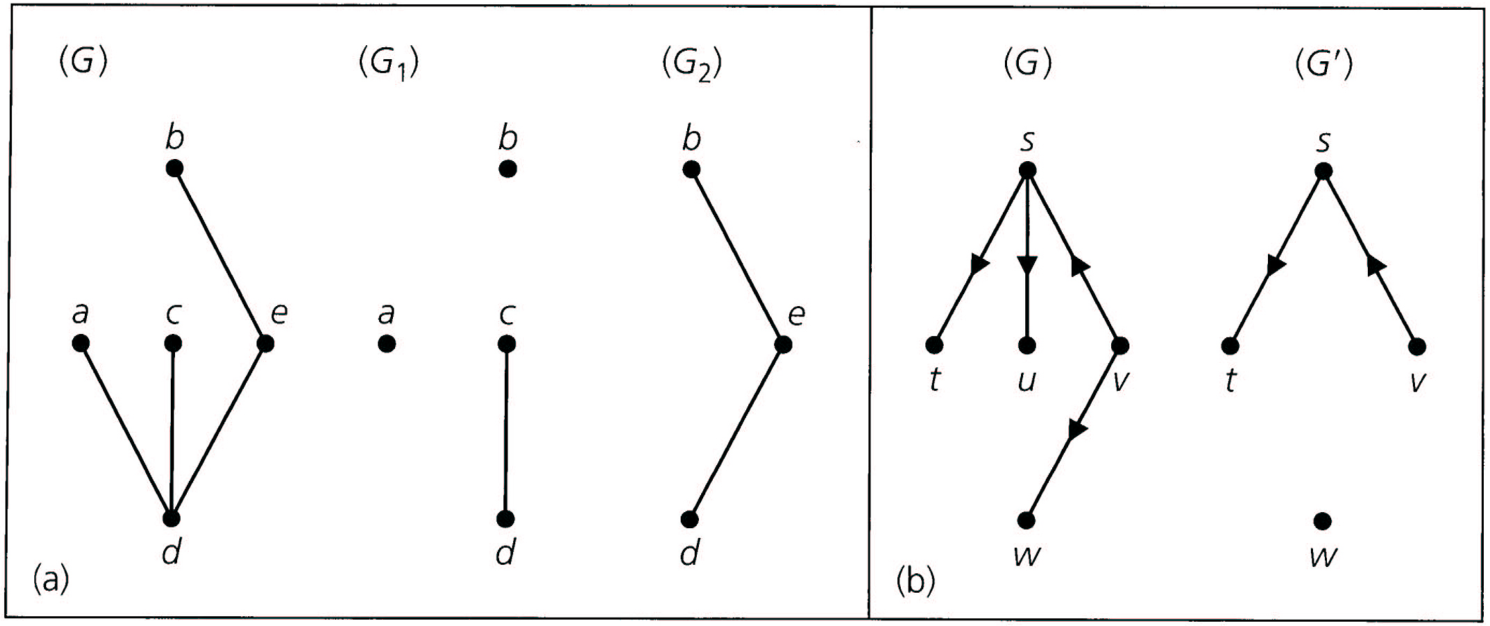

Figure 11.14

Spanning Subgraph

- For a graph $G=(V,E)$, let $G_1=(V_1,E_1)$ is a subgraph of G. If $V_1=V$, then G_1 is a spanning subgraph of G.

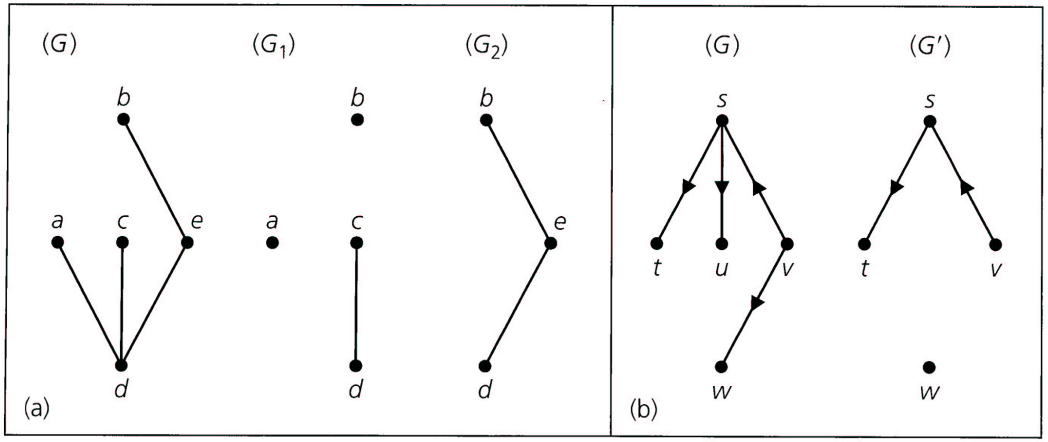

Figure 11.14

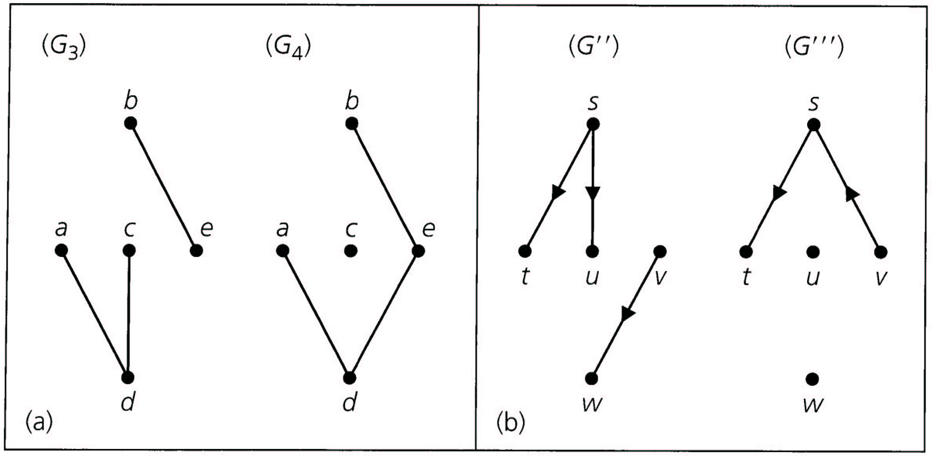

Figure 11.15

Induced Subgraph

- For a graph $G=(V,E)$, if $\varnothing \neq U \subseteq V$, the subgraph of G induced by U is the subgraph with all edges from G with the form: (a) (x,y) where $x, y \in U$ or (b) $\{x,y\}$ where $x, y \in U$. We denote this subgraph as <*U*>.
- A subgraph G' is called an induced subgraph if there exists a correspondent *U* such that *G'*=<*U*>.

Examples of Induced Subgraph

- Ex 11.5: In the figure, which subgraphs are induced subgraph?

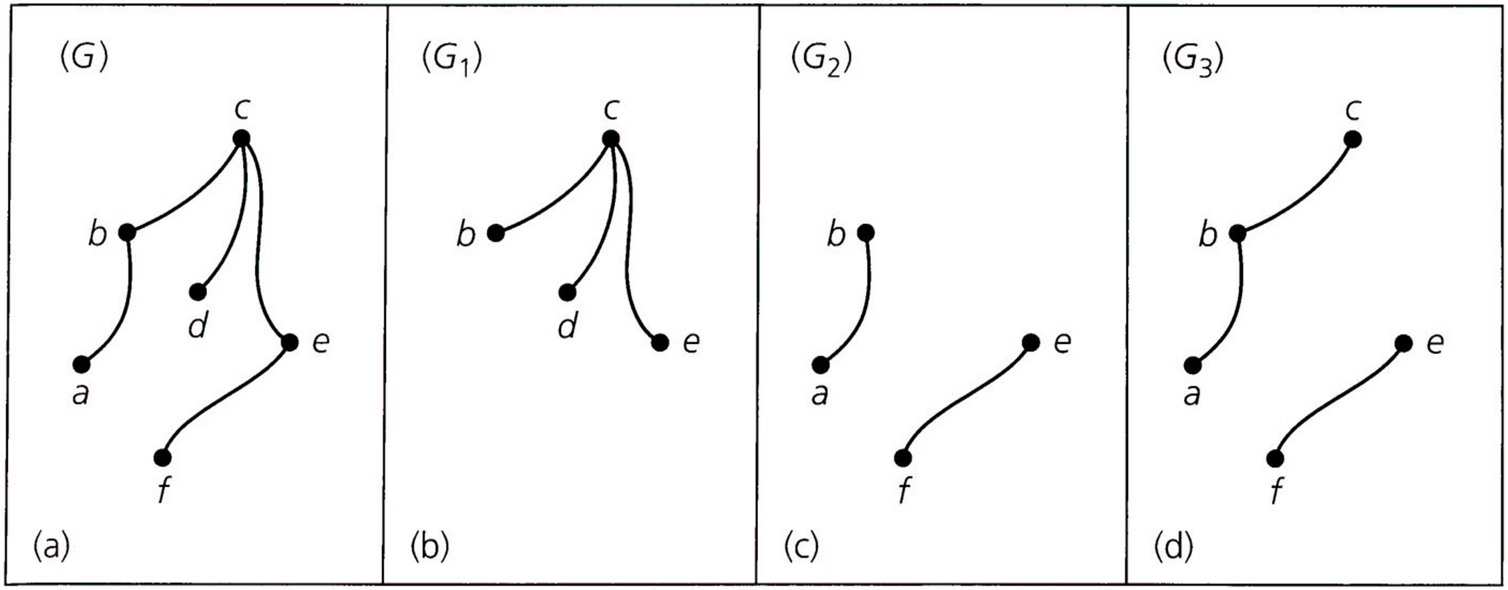

Figure 11.16

Removing a Vertex/Edge

- For a graph $G=(V,E)$, the subgraph $G\text{-}v$ has the vertex set $V_1=V-\{v\}$ and edge set E_1 containing all the edges in E that are not incident with v.
  - $G\text{-}v$ is the subgraph of G induced by V_1

- Subgraph $G\text{-}e=(V_1,E_1)$, where $E_1=E-\{e\}$ and $V1=V$.

Examples of Removing Vertexs/Edges

- Ex 11.6: What is G-c=<V-{c}>, G-e, and (G-b)-f

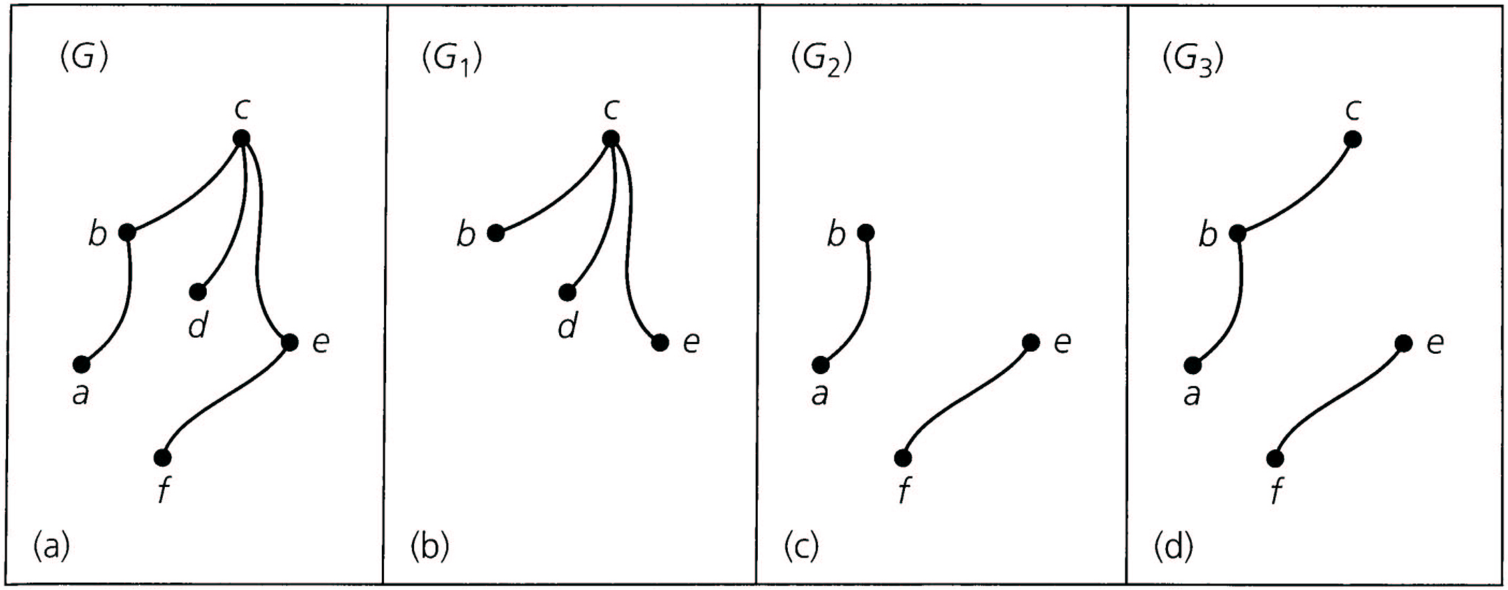

Figure 11.16

Complete Graph

- Let V be a set of n vertices. The complete graph on V, denoted K_n, is a loop-free undirected graph, where for any two distinct vertices a and b, there is an edge $\{a,b\}$.

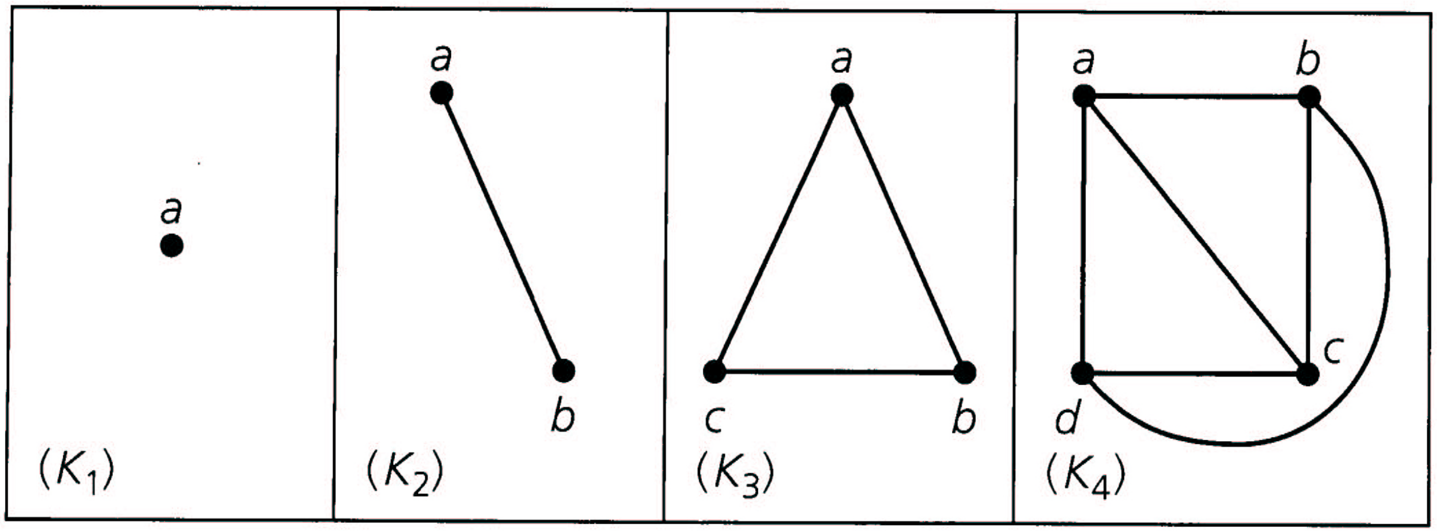

Figure 11.18

Complement

- For a graph G=(V,E) with n vertices. The complement of G, denoted $\bar{G}$, is the subgraph of K_n consisting of the n vertices of G and all edges that are not in G.
  - What is $\bar{K}_n$? $\bar{K}_n$ Is called a null graph.

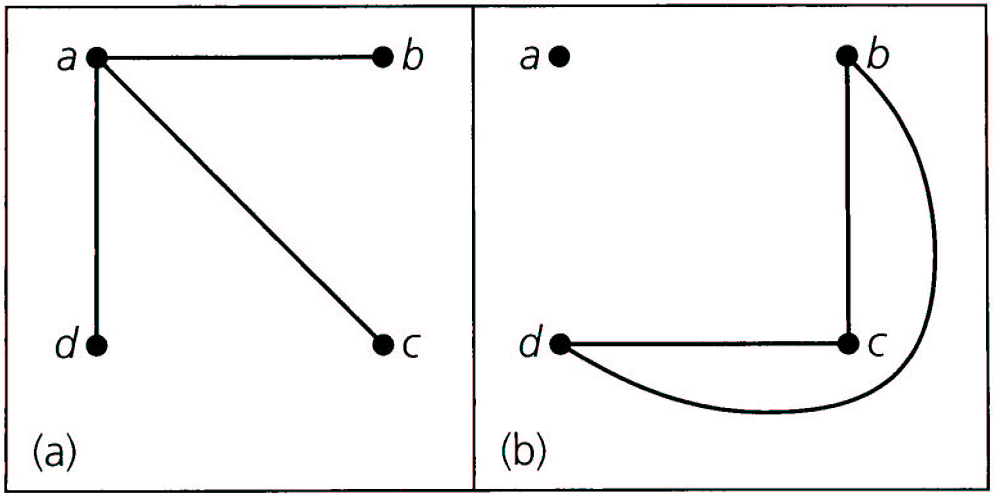

Figure 11.19

A Puzzle Problem

- Ex 11.7: Instant Insanity is played with four cubes. The objective of this game is to place the four cubes in a column, such that all four colors appear on each of the four sides.
- How can we keep track of the colors?

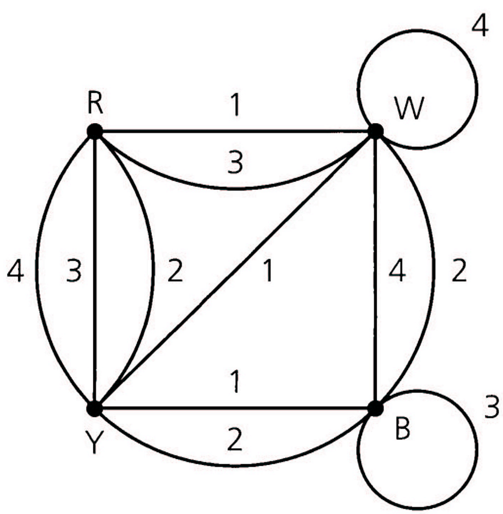

Figure 11.21

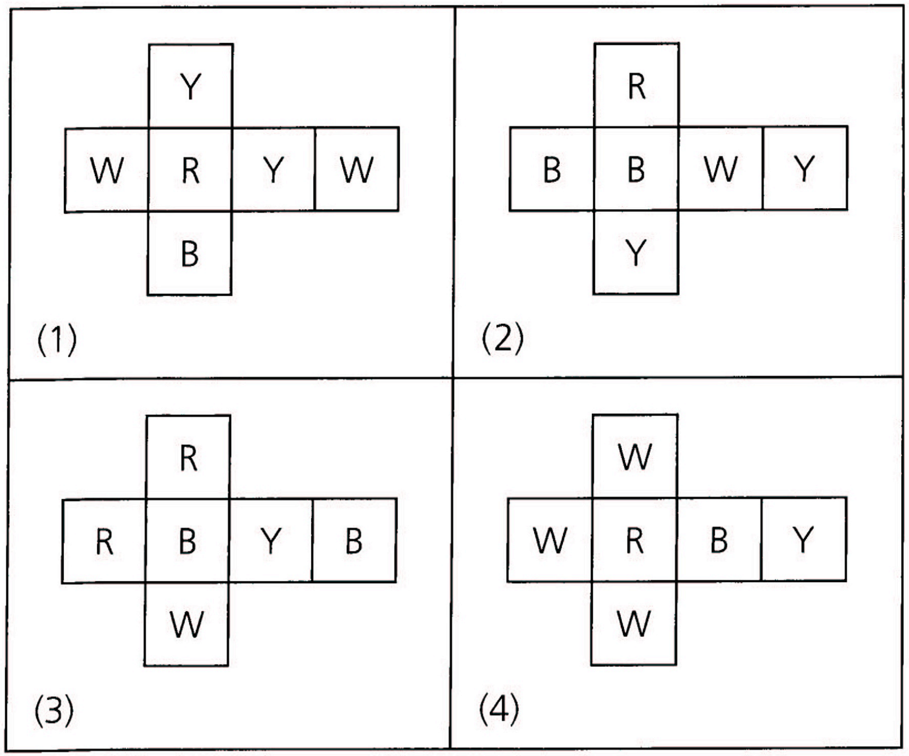

Figure 11.20

A Puzzle Problem (cont.)

- What we are looking for are two subgraphs with four vertices, so that each color appears only once, and each cube appears only once.

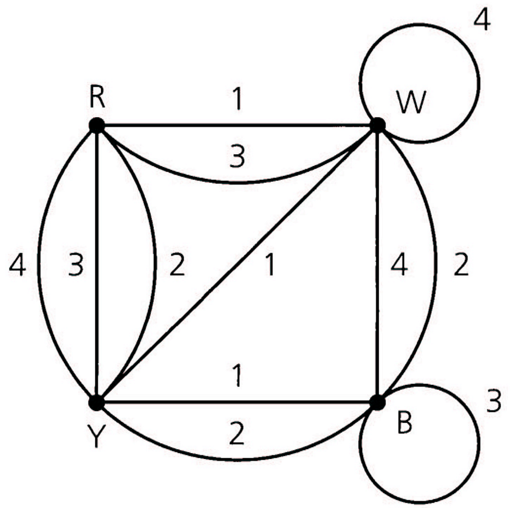

Figure 11.21

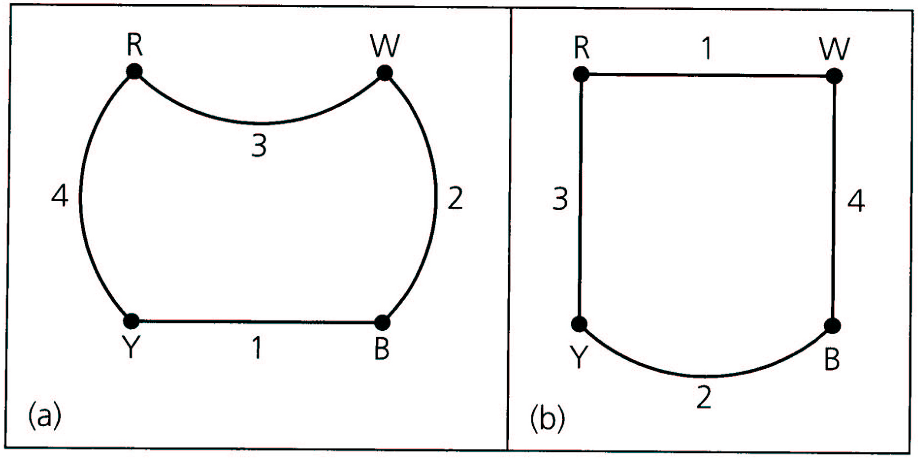

Figure 11.22

A Puzzle Problem (cont.)

- Generally, we construct a labeled multigraph, and try to find two subgraphs
  - Each subgraph contains all four vertices, and four labels
  - Each vertex is incident with two edges
  - No edge appears in both subgraphs

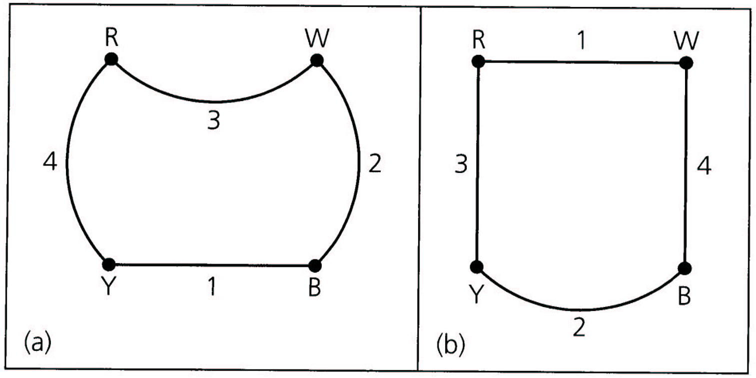

Figure 11.22

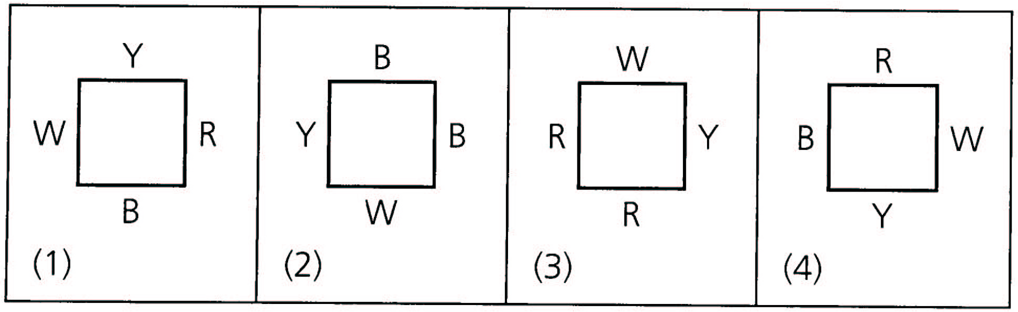

Figure 11.23

Isomorphism

- Let $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ be two undirected graph. A function $f : V_1 \to V_2$ is a graph isomorphism if
  - f is one-to-one and onto
  - For $\forall a, b \in V_1$, $\{a, b\} \in E_1$ iff $\{f(a), f(b)\} \in E_2$
- When such f exists, G_1 and G_2 are isomorphic graphs

Examples of Isomorphism

- What are the graph isomorphism function: (i) from (a) to (b), and (ii) from (c) to (d)

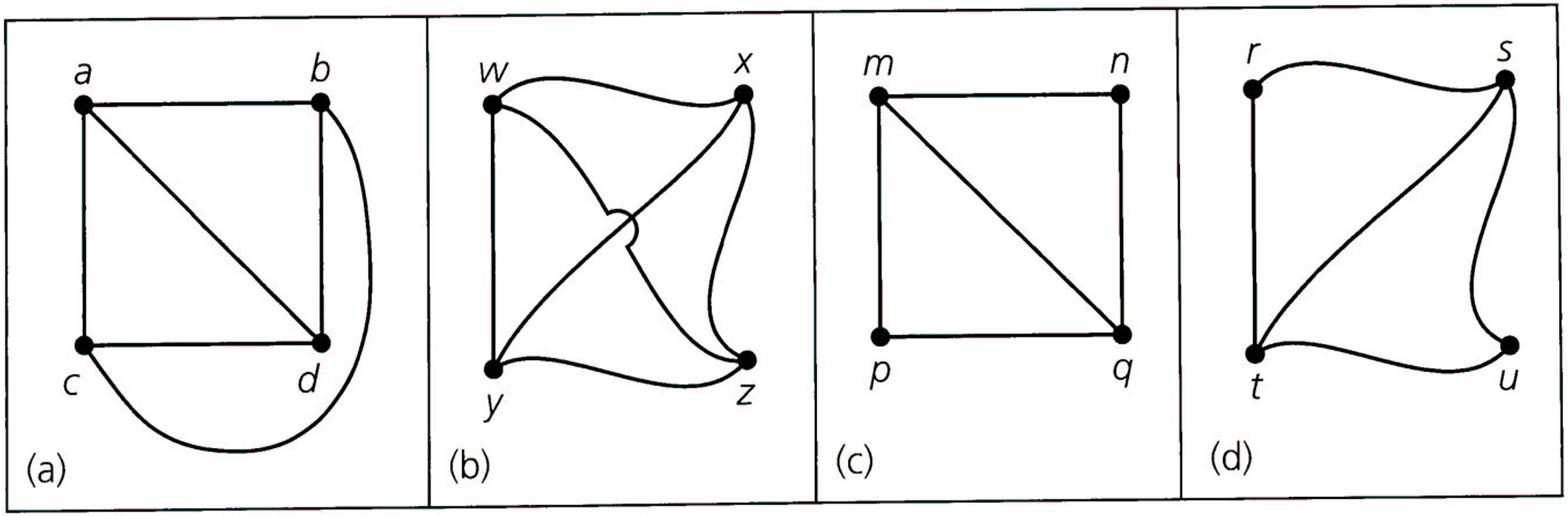

Figure 11.24

A Real Example

- Ex 11.8: Are the isomorphic?
- Observe: a→f→i→d→e→a, a cycle with length 5. Can we find a similar substructure in Fig. (b)?
- Verify the mapping:

a→ *q*
c→ *u*
e→ *r*
g→ *x*
i→ *z*
b→ *v*
d→ *y*
f→ *w*
h→ *t*
j→ *s*

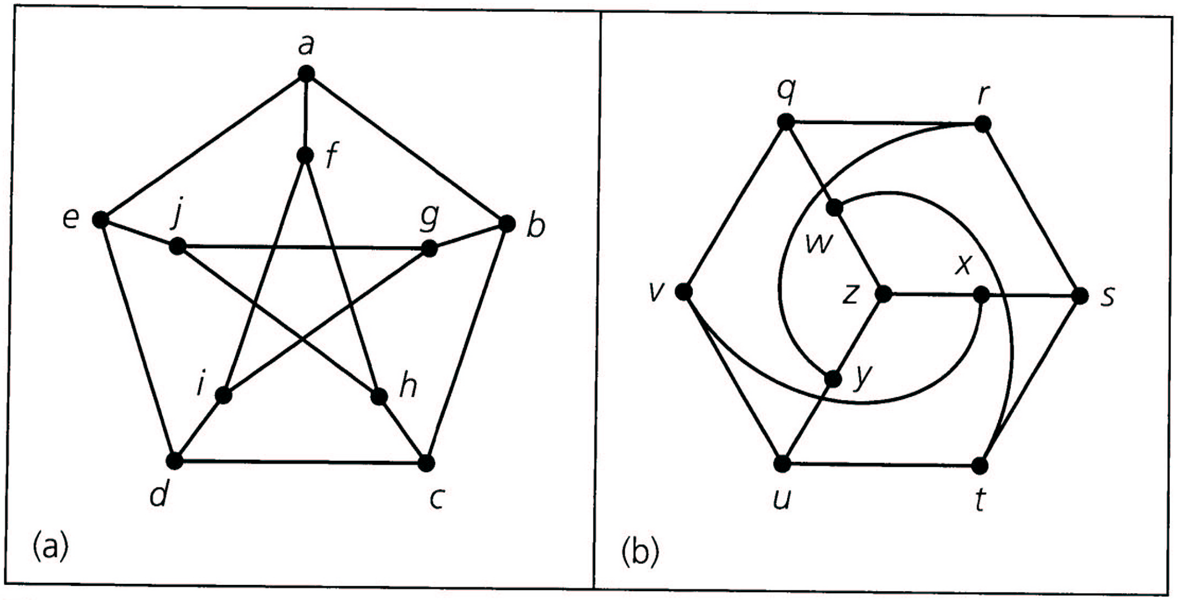

Figure 11.25

A Real Example

- Ex 11.9: Are the isomorphic?
  - Same numbers of vertices and edges
  - Circuit includes all edges
  - Number of degree-4 vertices

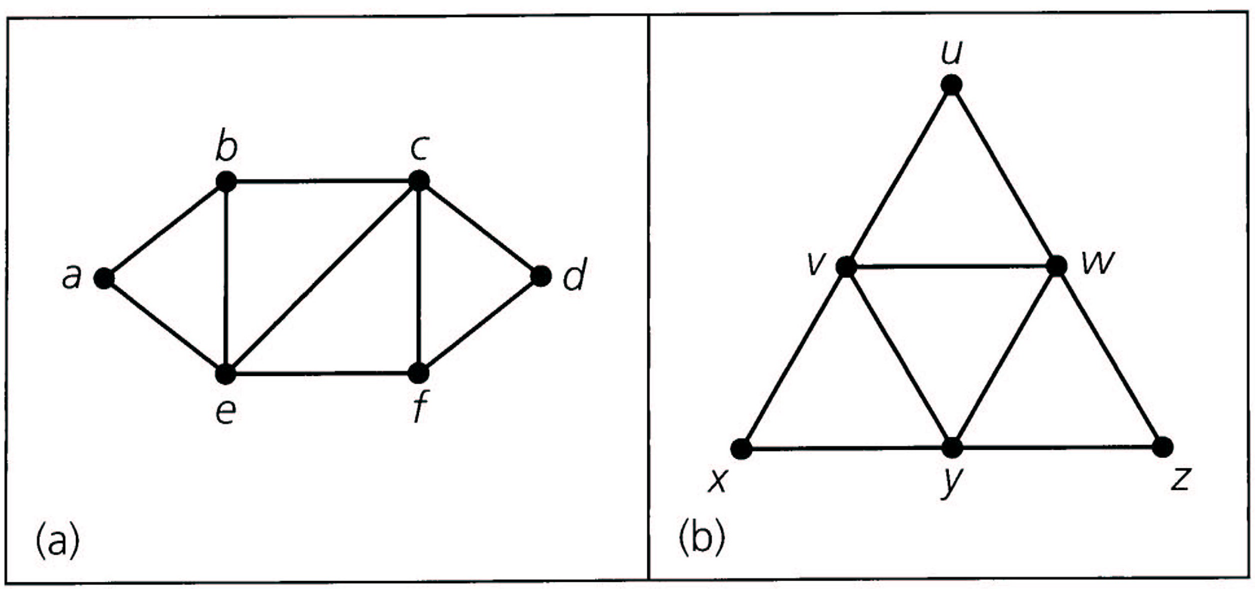

Figure 11.26

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Degree

- Let G be an undirected graph or multigraph. For each vertex v of G, the degree of v, $deg(v)$, is the number of edges into v. A loop is counted twice.
- Ex 11.10: What are the degrees of each vertex? We call a vertex with degree 1 as pendant vertex

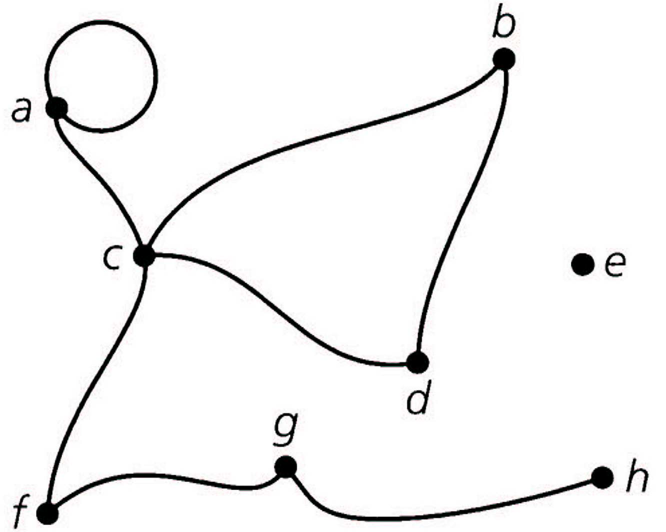


Figure 11.32

Number of Edges

- If G=(V,E) is an undirected graph or multigraph, we have $\sum_{v \in V} deg(v) = 2|E|$

- For any undirected graph or multigraph, the number of vertices of odd degree must be even

Regular Graphs

- An undirected graph, where each vertex has the same degree is called a regular graph. If deg(v)=k for all vertices v, the graph is called k-regular.
- Ex 11.11: Is it possible to have 4-regular graph with 10 edges?
  - $2|E|=20=4|V|$
- How about 4-regular graph with 15 edges?
  - $2|E|=30=4|V|$?

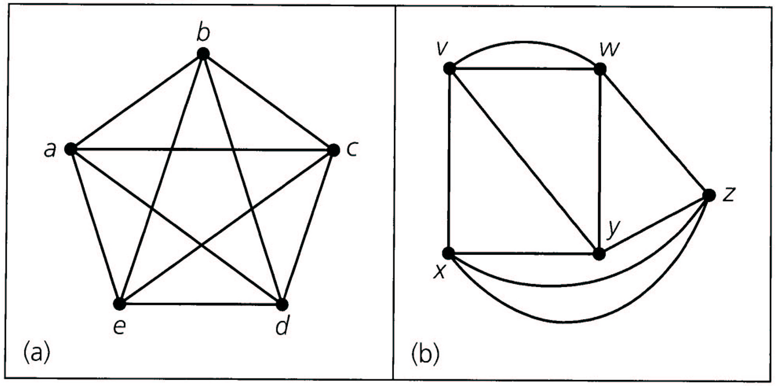


Figure 11.33

Hupercube

- Ex 11.12: To maximize the performance of a parallel computer, we need to build a network among all the processors
- Ideally in fully-mesh, fast but expensive
- Grid (or mesh) graph is cheaper, but may not scale well.

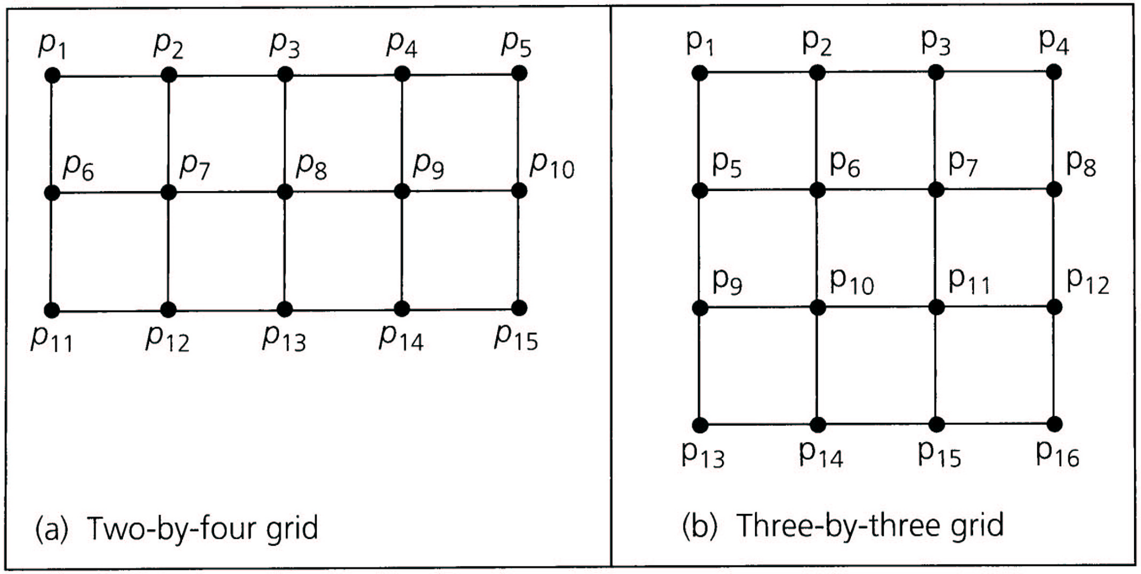


Figure 11.34

Hupercube (cont.)

- N-cube is denoted as Q_n, and has 2^n vertices. Each vertex is labeled by a n-bit sequence, ranges from 0 to 2^n-1.
- Two vertices are connected if they differ in 1 bit.

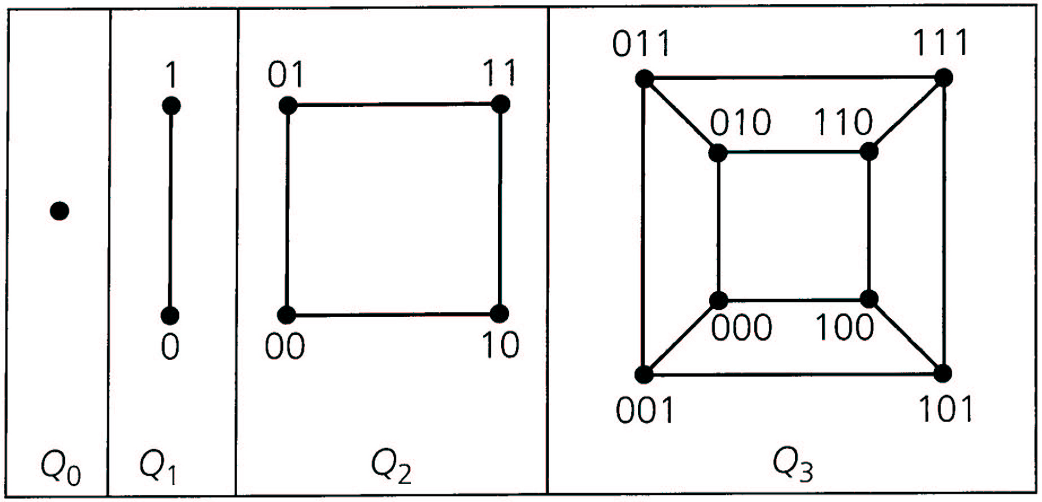


Figure 11.35

- Leads to shorter distance!

Hupercube (cont.)

- How about Q_5?

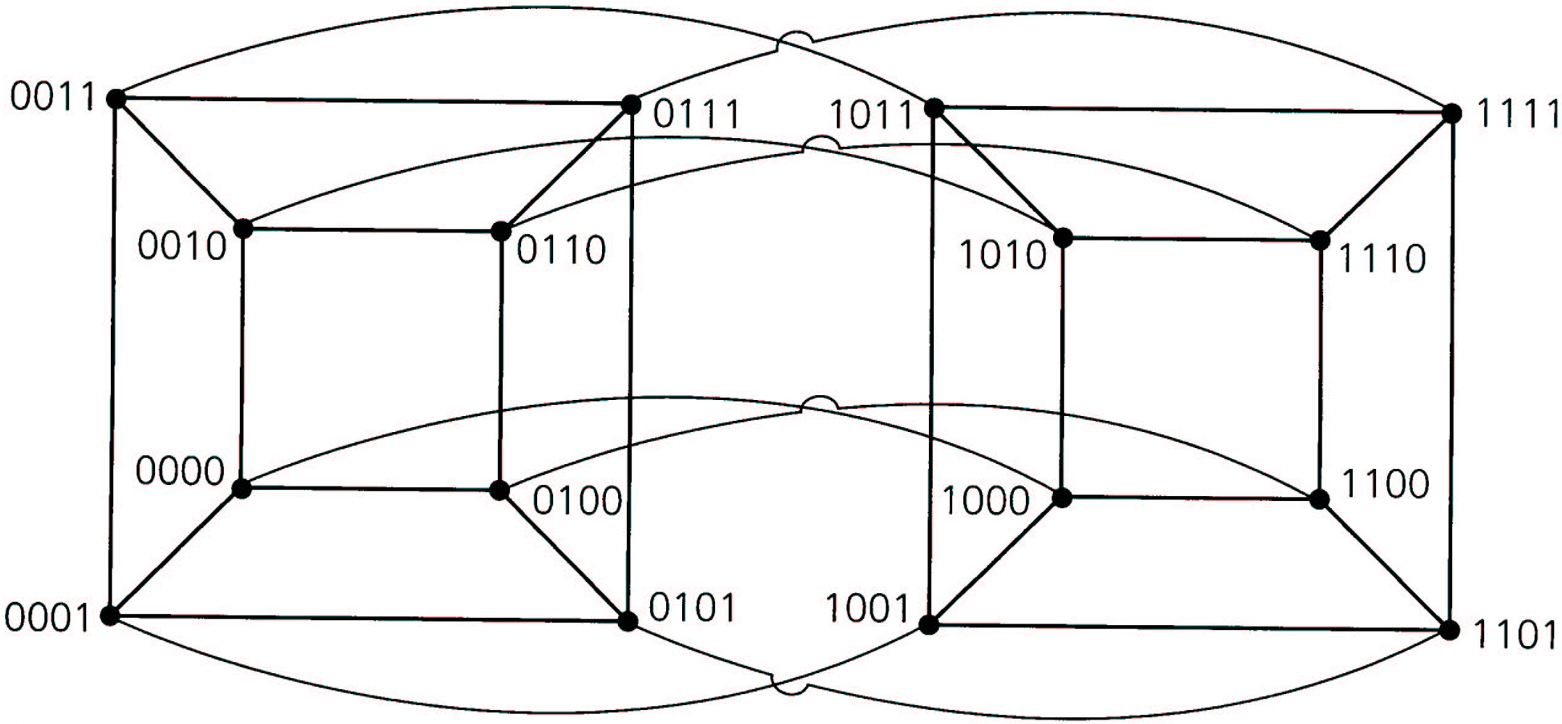


Figure 11.36

Seven Bridges of Konigsberg

- Ex 11.13: The city is divided into four sections. The residents spent Sundays trying to find a way to walk around the city so that they cross each bridge exactly once and return to the starting point.

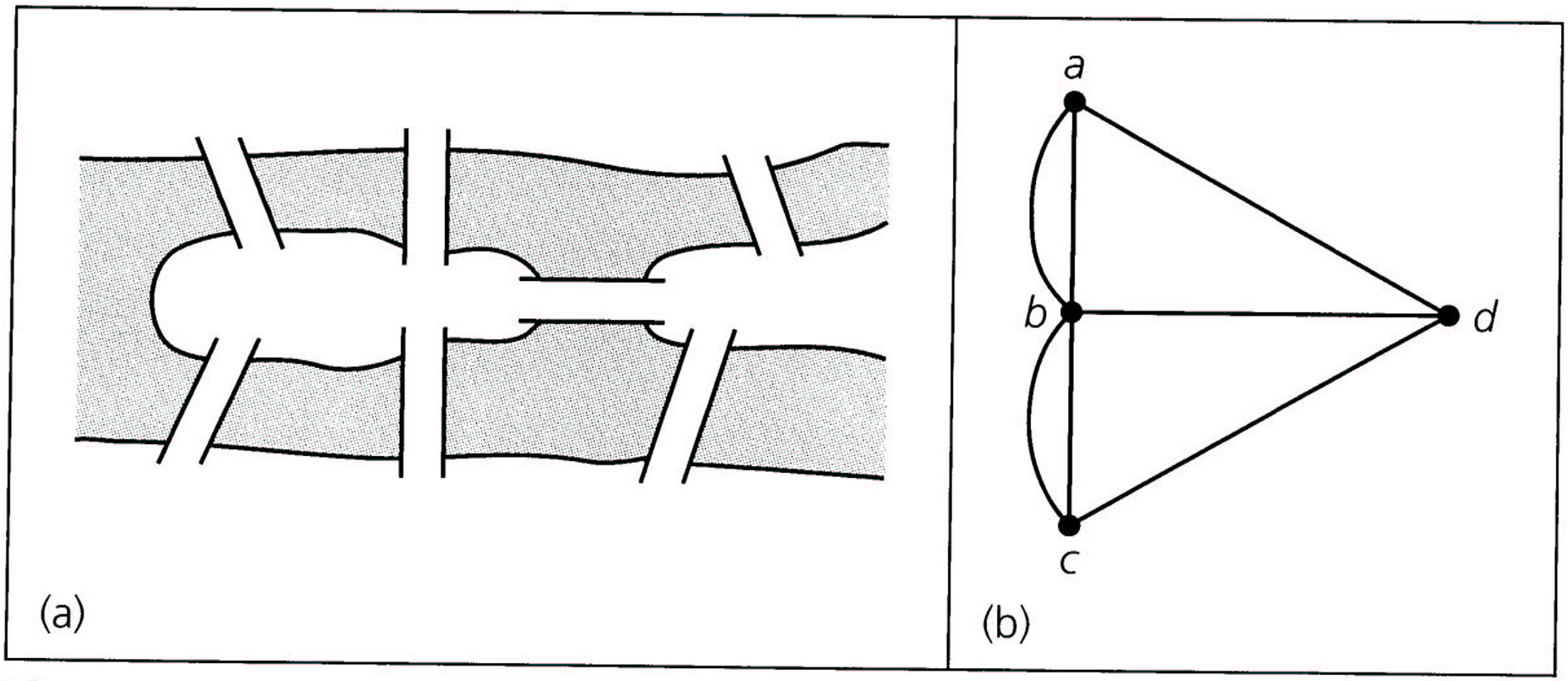


Figure 11.37

Euler Circuit

- Let G(V,E) is a graph with no isolated vertices. G has an Euler circuit if there is a circuit in G that traverses every edge of the graph exactly once.
  - For a trail going through each edge once, it is called Euler trail

- A graph (without isolated vertices) has an Euler circuit iff *G* is connected and every vertex in *G* has even degree.

Euler Circuit (cont.)

- A graph (without isolated vertices) contains an Euler trail iff G is connected and has exactly two vertices with odd degree.

- Do we have Euler circuit or trail?

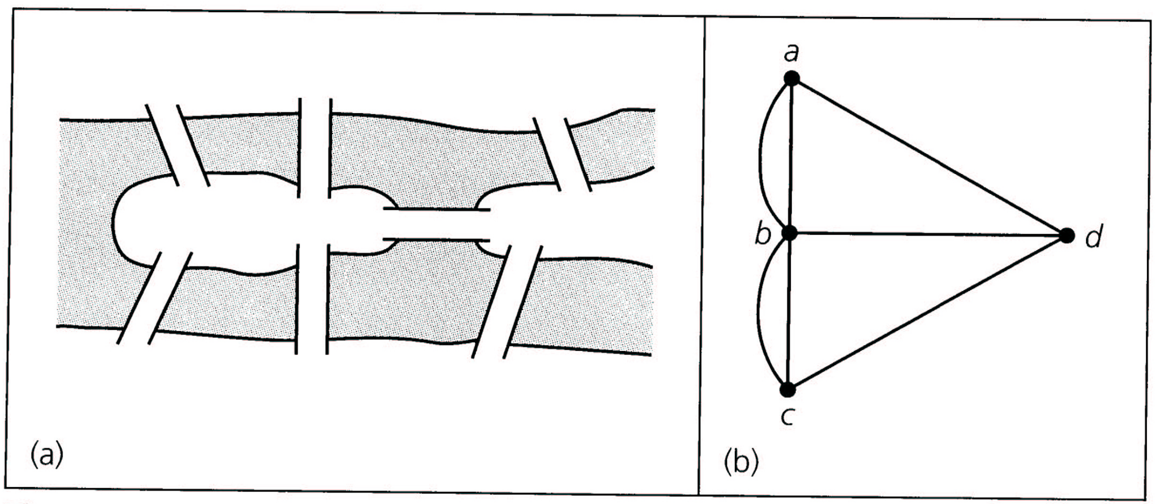


Figure 11.37

Incoming and Outgoing Degrees

- Let $G(V,E)$ be a directed graph or multigraph. For each v
  - Incoming (or in) degree, $id(v)$ is the number of edges incident into v
  - Outgoing (or out) degree, $od(v)$ is the number of edges incident from v
- Each loop counts as one incoming and one outgoing degrees.

Directed Euler Circuit

- Let $G(V,E)$ be a directed graph with no isolated vertices. The graph G has a directed Euler circuit iff G is connected and $id(v)=od(v)$ for all vertex v.

Example of Directed Euler Circuit

- Consider a rotating drum

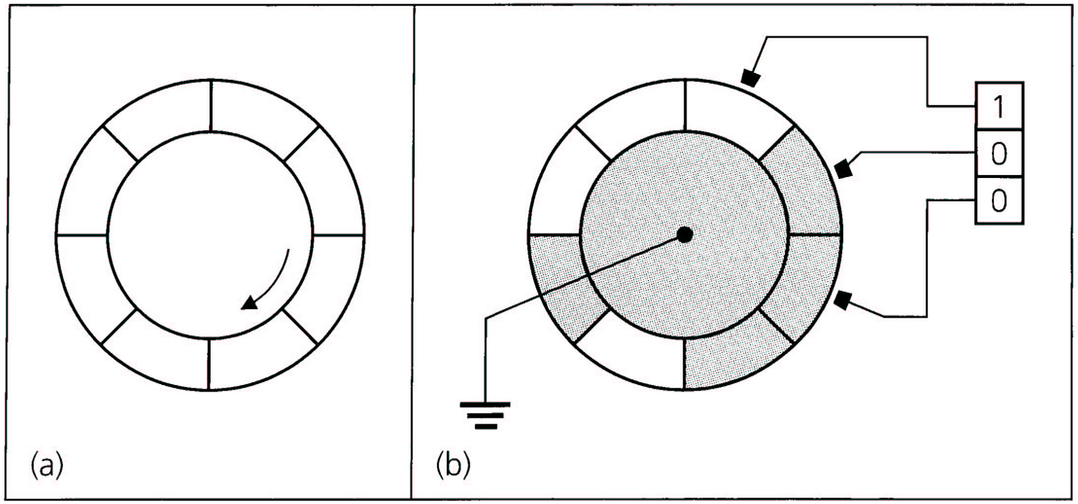


Figure 11.39

- Can we represent all 3-digital numbers by turning the drum?

Example of Directed Euler Circuit (cont.)

- Construct a directed graph with V={00,01,10,11} and (b_1b_2,b_2b_3) in E if b_1b_2 and b_2b_3 are in V.
- Since each *v* has *id*(*v*)=*od*(*v*), there is a directed Euler circuit 10→00→00→01→10→01→11→11

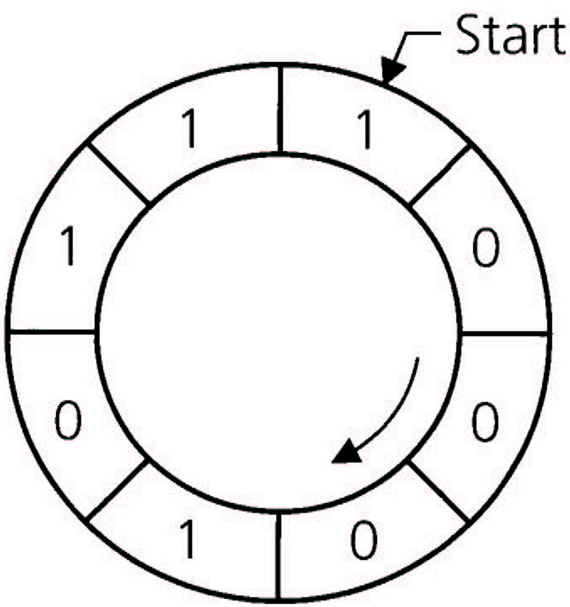


Figure 11.41

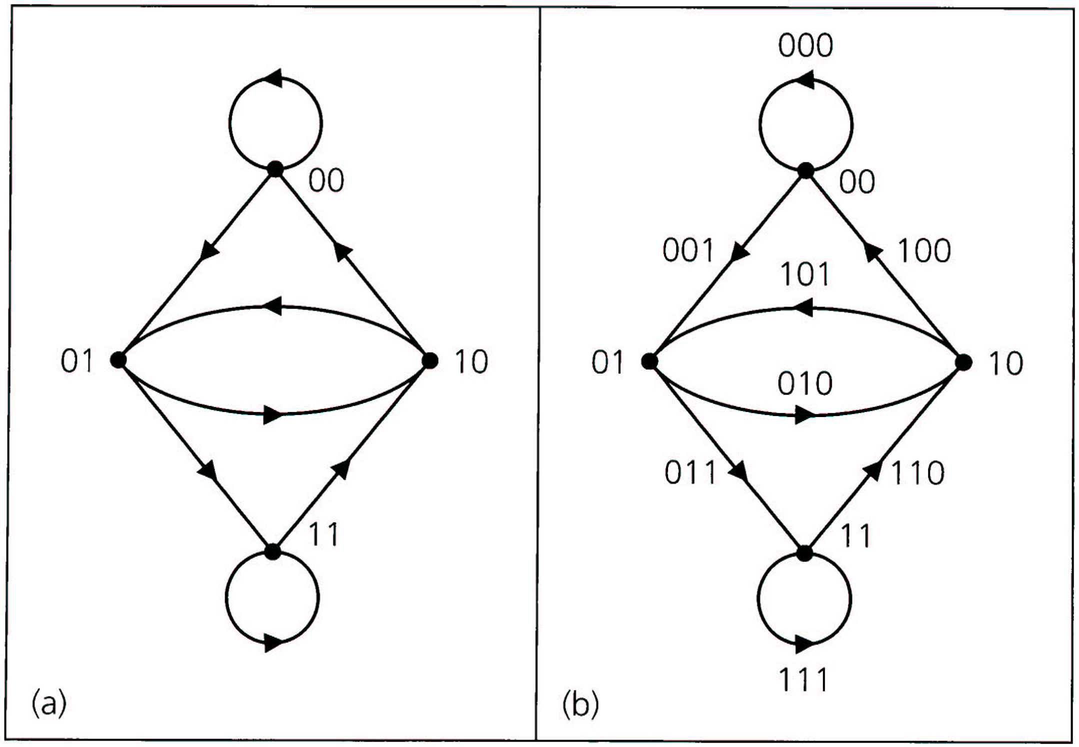


Figure 11.40

Outline

Planar Graphs

- A graph (or multigraph) G is planar if G can be drawn in the plane, where all its edges only intersect at its vertices.
  - We call such a drawing of G as an embedding of G in the plane
- Ex 11.15: Are the following graphs planar or nonplanar?

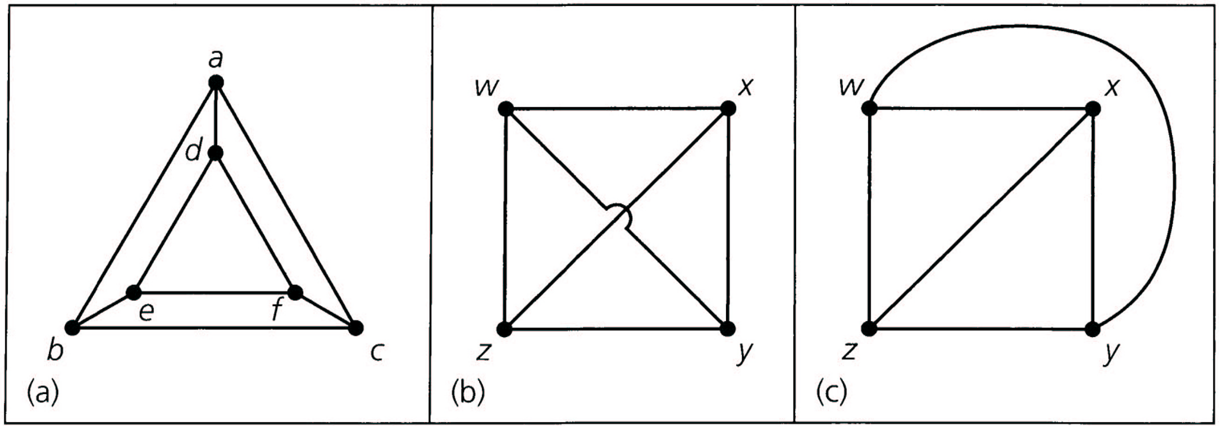


Figure 11.47

Complete Graphs and Planar

- K_1, K_2, K_3, and K_4 are all planar.
- Ex 11.16: Is K_5 Planar?

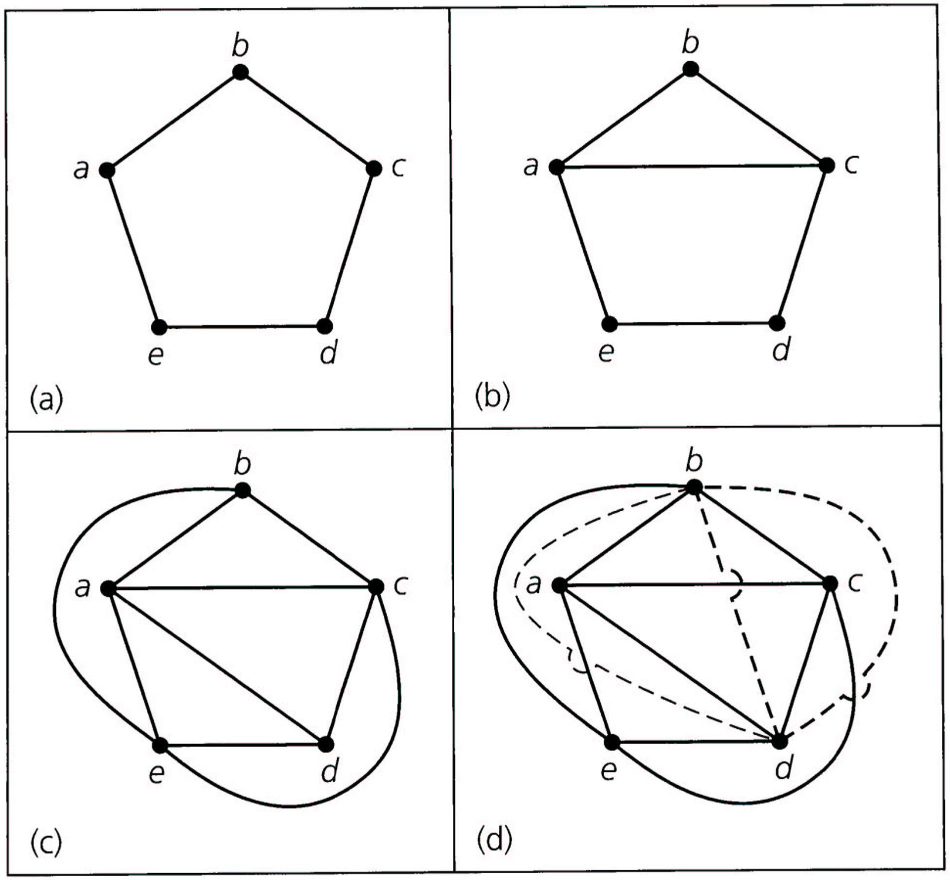


Figure 11.48

Bipartite Graph

- $G=(V,E)$ is bipartite if V can be divided into a partition of V_1 and V_2, and each edge is in the form of $\{a,b\}$ where a is in V_1 and b is in V_2.

- If every vertex in V_1 is connected to every vertex in V_2, then we have a complete bipartite graph. This graph is denoted as $K_{m,n}$, where $|V_1|=m$ and $|V_2|=n$.

Hypercubes are Bipartite

- Ex 11.17: If we divide the vertices based on the even/odd numbers of one's in the binary strings

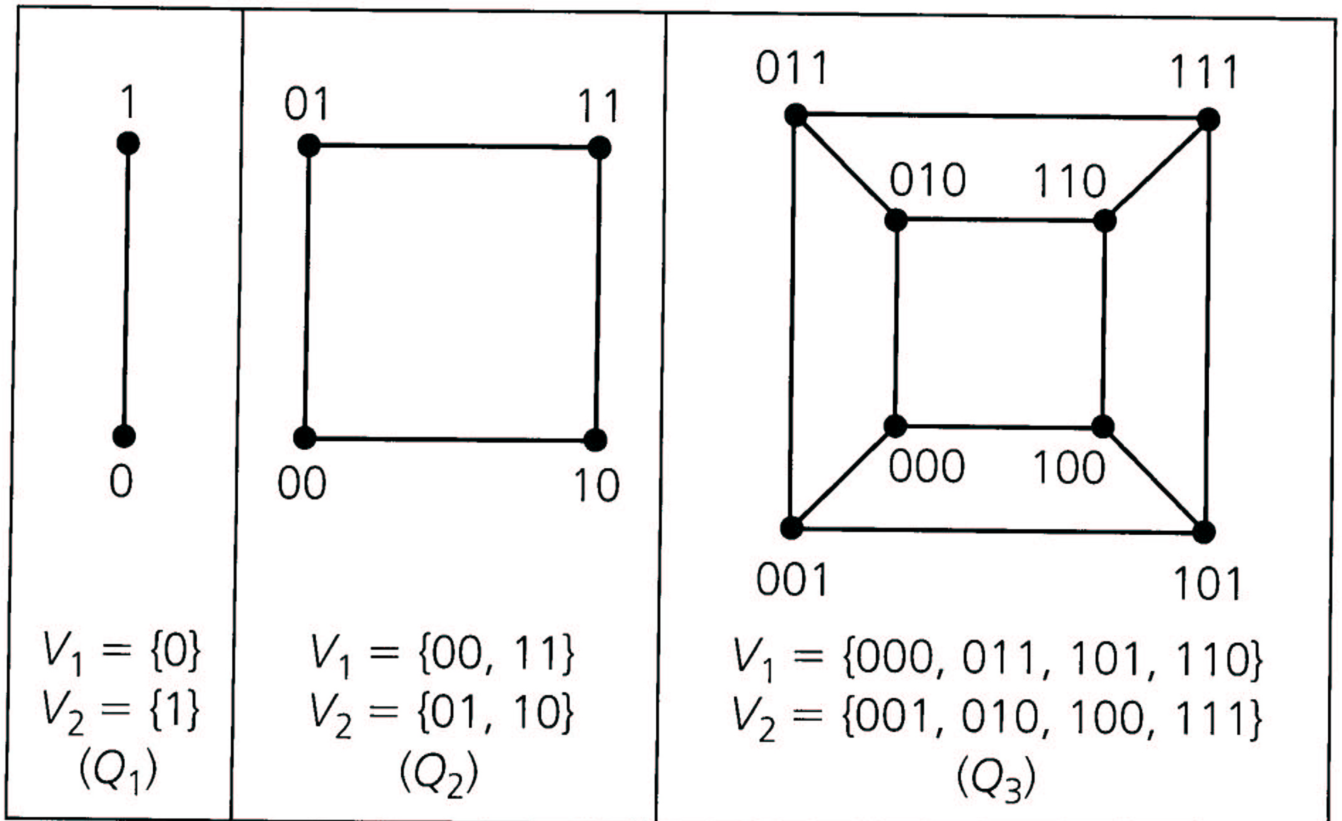


Figure 11.49

Utility Graph

- The left bipartite graph can be expanded to complete bipartite graph $K_{2,3}$, which is planar. How?
- The right bipartite graph can also be expanded to $K_{3,3}$. But can it be planar?
- Preview: K_5 and $K_{3,3}$ are the source of nonplanar.

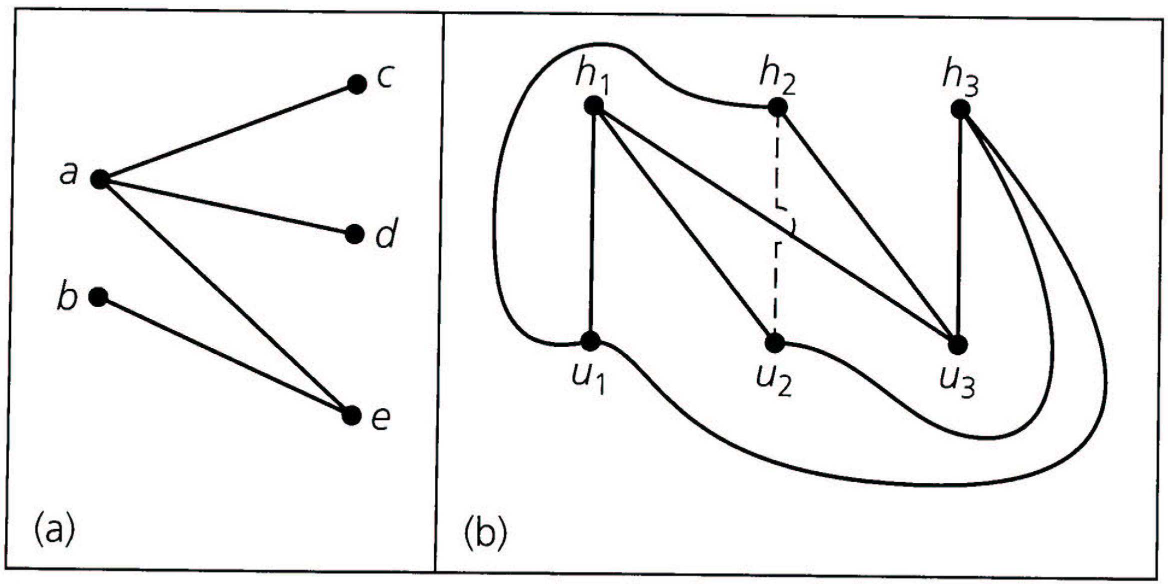


Figure 11.50

Elementary Subdivision

- $G=(V,E)$ is a loop-free undirected group with nonempty E. An elementary subdivision of G leads to removing an edge $e=\{u,w\}$ and adding the edges $\{u,v\}$ and $\{v,w\}$ where v was not in V.

- Two loop-free undirected graphs G_1 and G_2 are homeomorphic if they are isomorphic or if they can both be obtained from the same graph H by a sequence of elementary subdivisions.

Some Examples

- G=(V,E) is a loop-free undirected group with |E| >=1. Any G' derived from G by an elementary subdivision has |V|+1 vertices and |E|+1 edges
- Ex 11.18: G_1 is derived from G_2 by an elementary subdivision. G_1, G_2, and G_3 are mutually homeomorphic. Can we get G_2 from G_1?

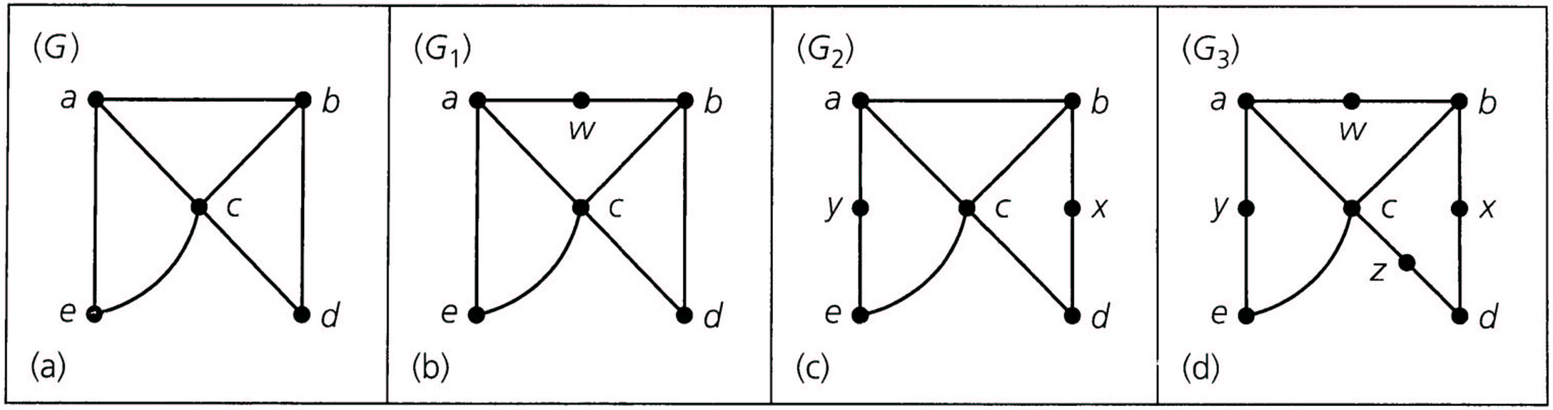


Figure 11.51

Homeomorphic

- Can be thought as isomorphic except vertices of degree 2

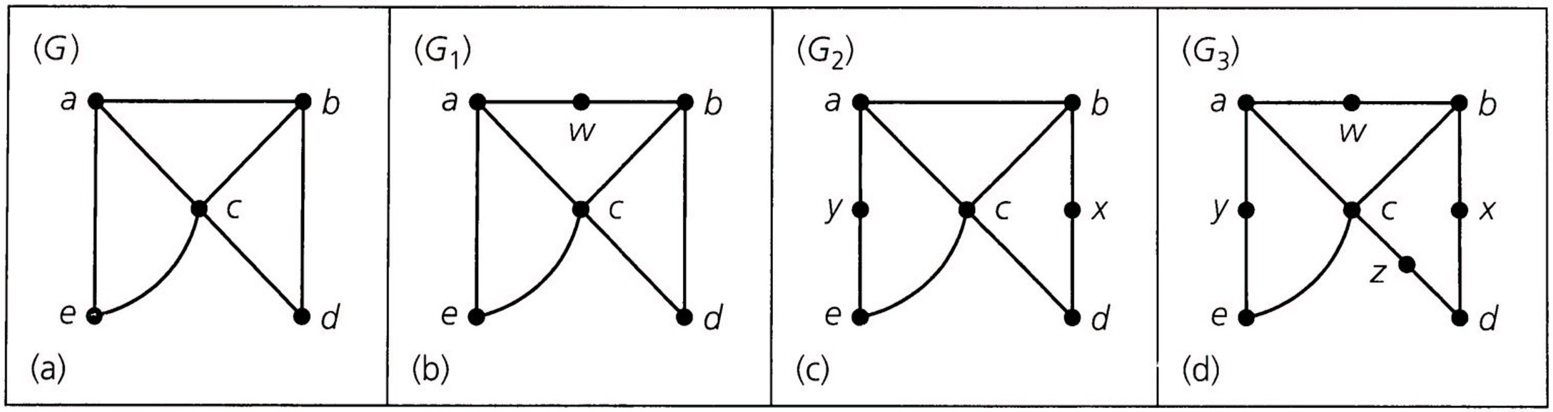


Figure 11.51

- If two graphs are homeomorphic, they are either both planar or both nonplanar

Kuratowski's Theorem

- A graph is nonplanar iff it contains a subgraph that is homeomorphic to either K_5 or $K_{3,3}$

- Ex 11.19 (a): Petersen graph contains a subgraph that is homeomorphic to $K_{3,3,}$ and thus is nonplanar

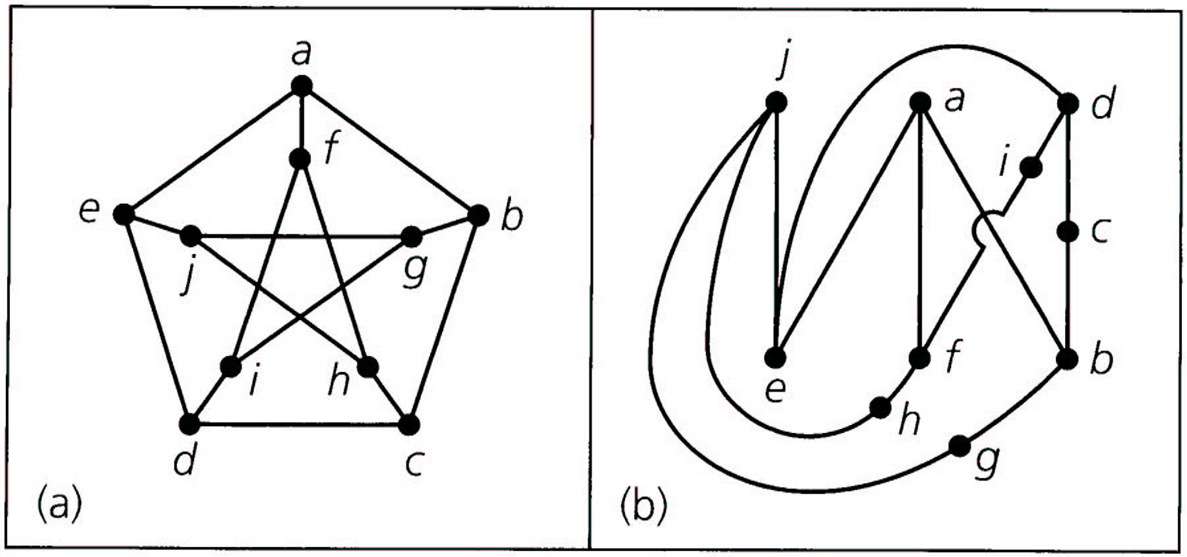


Figure 11.52

More Examples

- Ex 11.19 (b):
  - Q3 is planar
  - The complement of Q3 looks like nonplanar
  - H is a subgraph of the complement of Q3, which shows the complement of Q3 is nonplanar

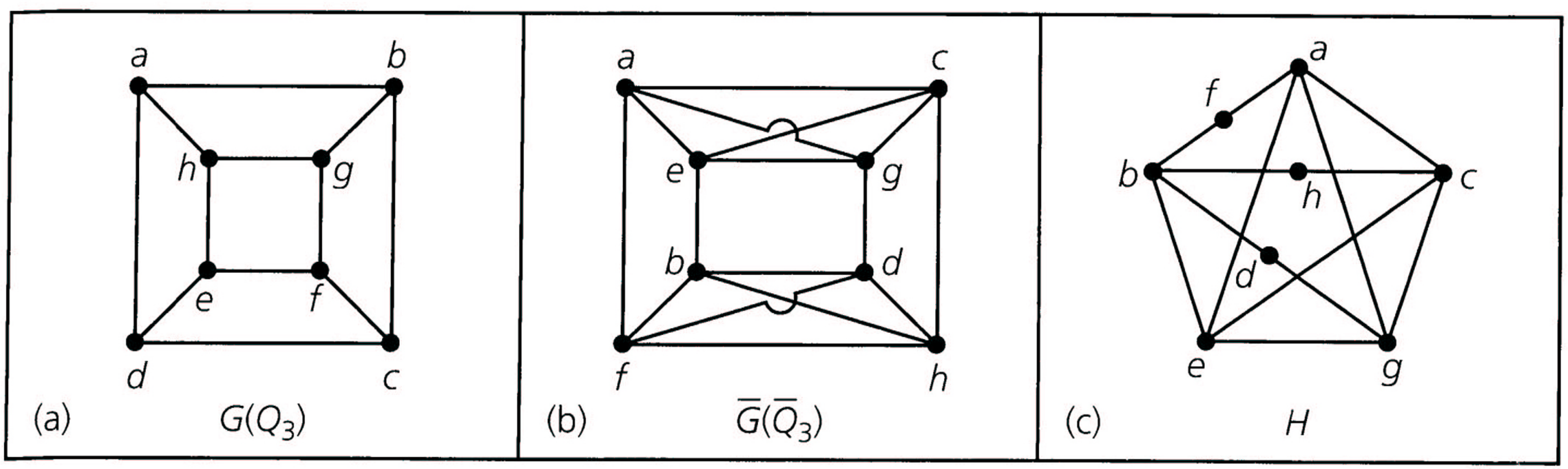


Figure 11.54

Regions

- Each planar embedding of a graph defines regions
  - Infinite regions
- Nonplanar graphs' regions are not defined

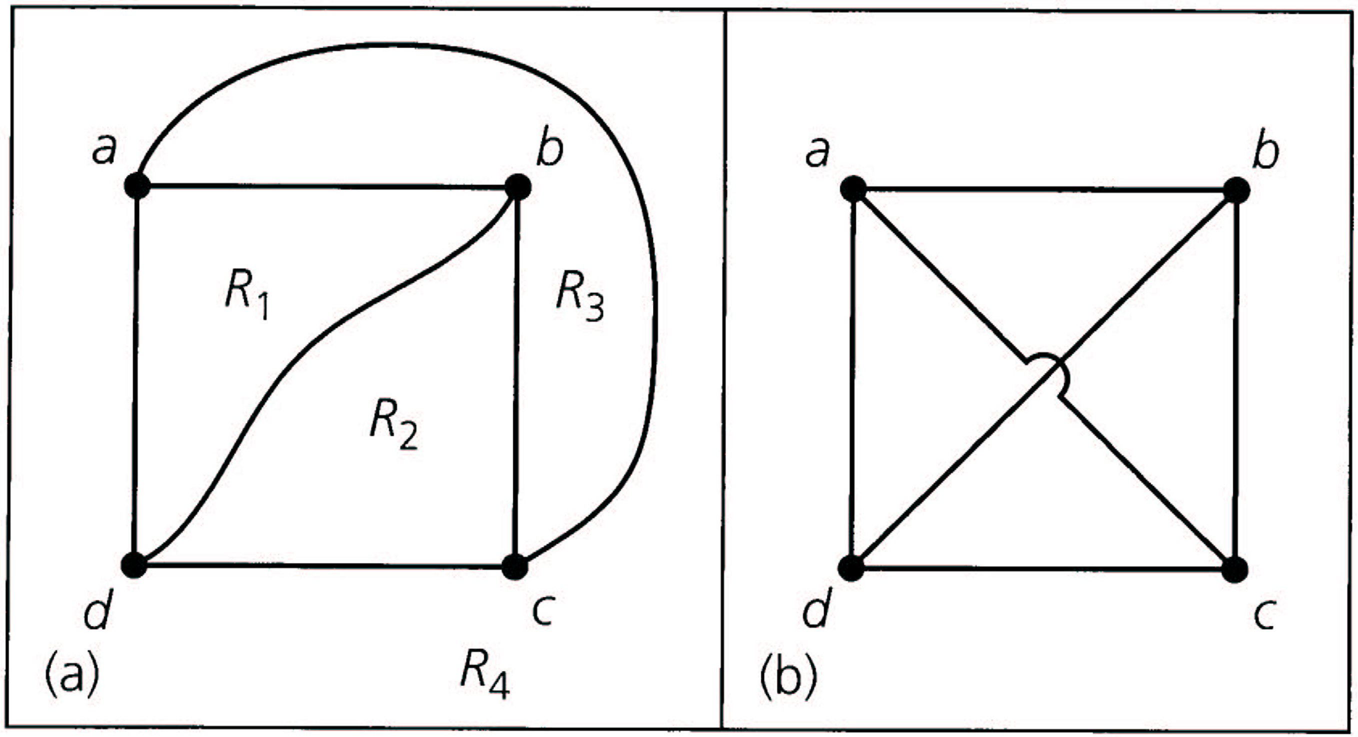


Figure 11.55

Euler's Theorem

- $G=(V,E)$ is a connected planar graph or multigraph, where $|V|=v$ and $|E|=e$. Let r be the number of regions. Then $v-e+r=2$
- Proved by induction: see text for details.

Degree of a Region

- Degree of a region R, deg(R), is the number of edges traversed in a shortest closed walk about the edges in the boundary of R.
- Ex:
  - $\deg(R_1)=5$, $\deg(R_2)=3$, $\deg(R_3)=3$, $\deg(R_4)=7$
  - $\deg(R_5)=4$, $\deg(R_6)=3$, $\deg(R_7)=5$, $\deg(R_8)=6$

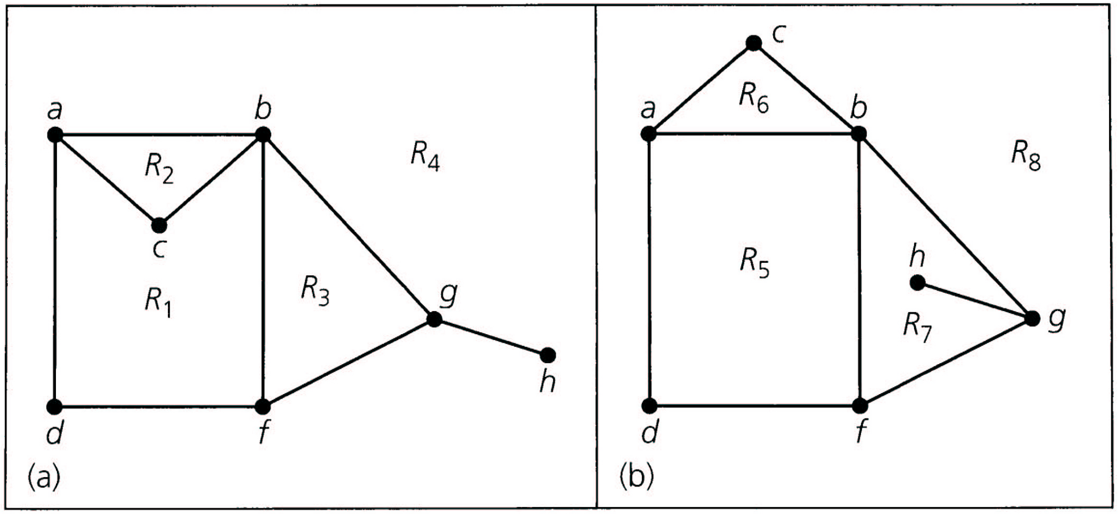


Figure 11.58

Corollary

- G=(V,E) is a loop-free connected planar graph with |V|=v, |E|=e>2, and r regions. Then, 3r<=2e and e<=3v-6
  - One way
- Ex 11.20: K_5 has 5 vertices and 10 edges. Show it is nonplanar
- Ex 11.21: $K_{3,3}$ has 6 vertices and 9 edges. Can we say it is planar by 9<=12? NO!
  - Bipartile graphs don't have triangle subgraph. So each region has degree >= 4. This leads to a contradiction to Euler's theorem

Platonic Solids

- Ex 11.22: Platonic Solids: Solids with all faces are congruent and all angles are equal.
- Tetrahedron: four faces, each face is a equilateral triangle.
- Cube: the planar graph has six regions, where three regions meets at each vertex

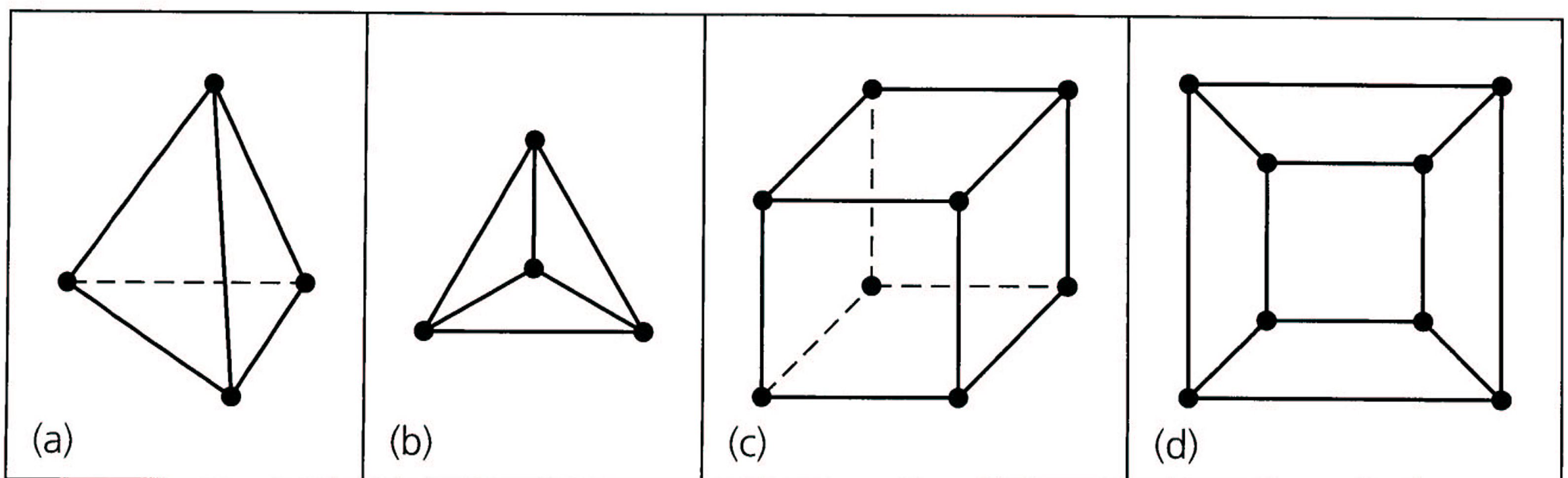


Figure 11.59

Platonic Solids (cont.)

- General case: v=|V|, e=|E|, r=number of planar regions determined by (V,E), m=number of edges at the boundary of each region, n=number of regions meet at each vertex
  - 2e=mr ← each edge is next to two regions
  - 2e=nv ← counting the endpoints
- Applying Euler's theorem, we have $(m-2)(n-2)<4$
- There are only five cases!
  - m=n=3 (thtrahedron), m=4,n=3 (cube)

Platonic Solids (cont.)

- Octahedron: m=3, n=4
- Dodecahedron: m=5, n=3
- Icosahedron: m=3, n=5

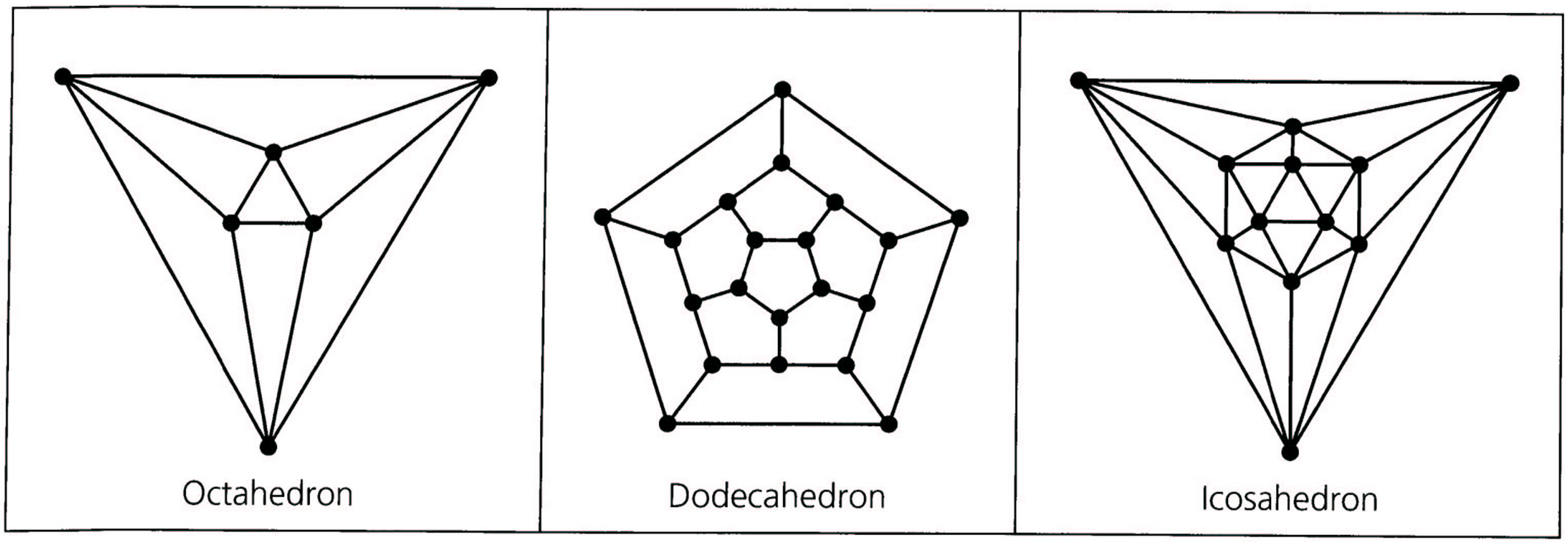


Figure 11.60

Dual Graph

- To construct a dual graph G^d with G
  - Place a vertex inside each region of G, including the infinite one
  - Draw an edge in G^d connecting the two regions sharing an edge in G
  - For an edge of G that is traversed twice in the closed walk around a region, draw a loop at the vertex in G^d

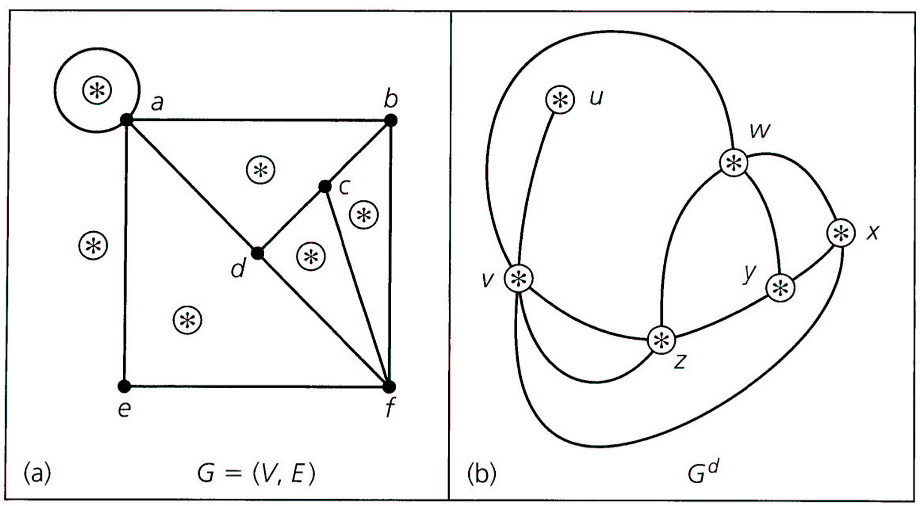


Figure 11.61

Cut-Set

- For an undirected graph or multigraph G=(V,E), a subset E' of E is called a cut-set of G is by removing the edges (but not the vertices) in E' from G, we have k(G) < k(G'), where G'=(V,E-E'). Moreover, if we remove any other proper subset E'' of E', we have k(G)=k(G''), where G''=(V,E-E'')

- For a cut-set with only one edge, we call it a bridge

Example of Cut-Sets

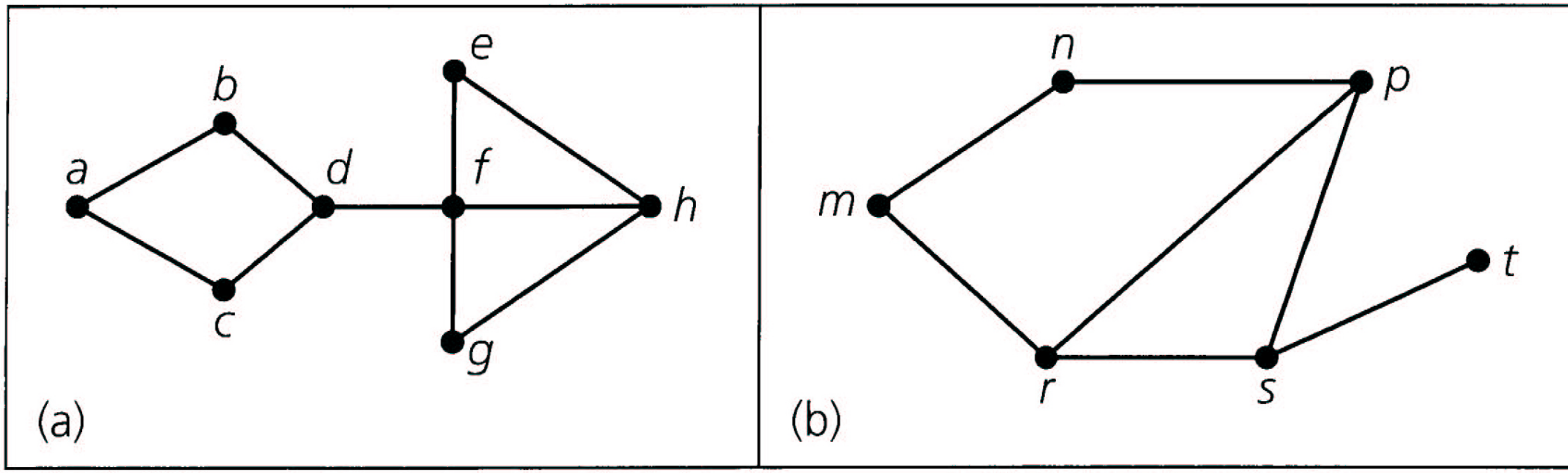


Figure 11.62

Some Observations

- 6,7, and 8 forms a cycle in G. What if we remove 6^*, 7^*, and 8^* in G^d? → G^d becomes disconnected!
  - A cut-set in G^d
- Check 2,4,and 10 in G?

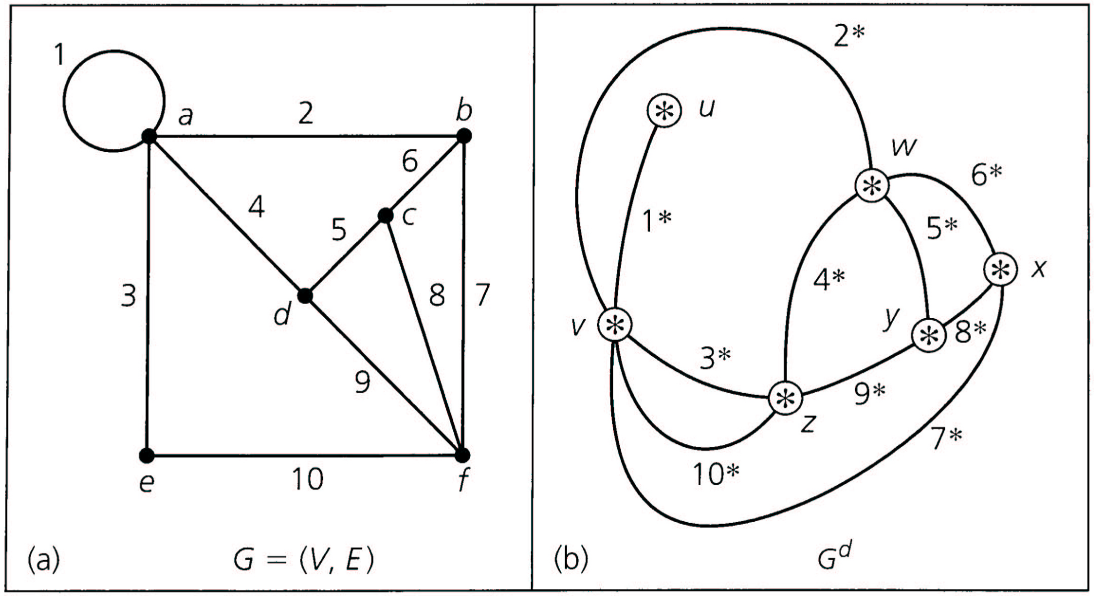


Figure 11.63

- More observations can be found on page 551

Map Coloring Problem

- Ex 11.24: A dual graph without the infinite region.
- Mapmaker's problem: How can we color the regions so that no two adjacent regions share the same color?

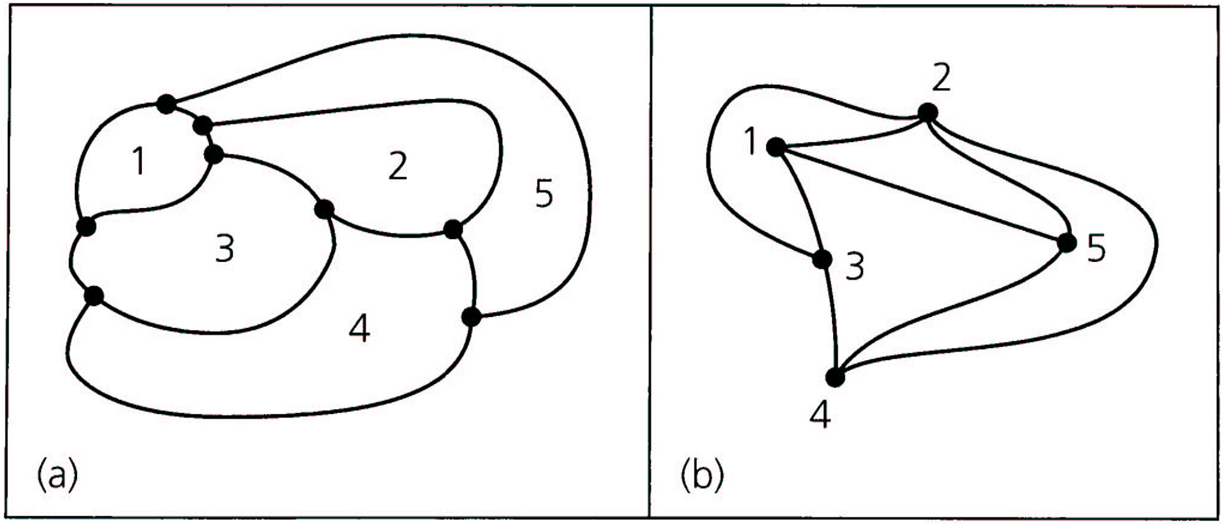


Figure 11.64

A Real Application

- Ex 11.25: A network with 9 switches

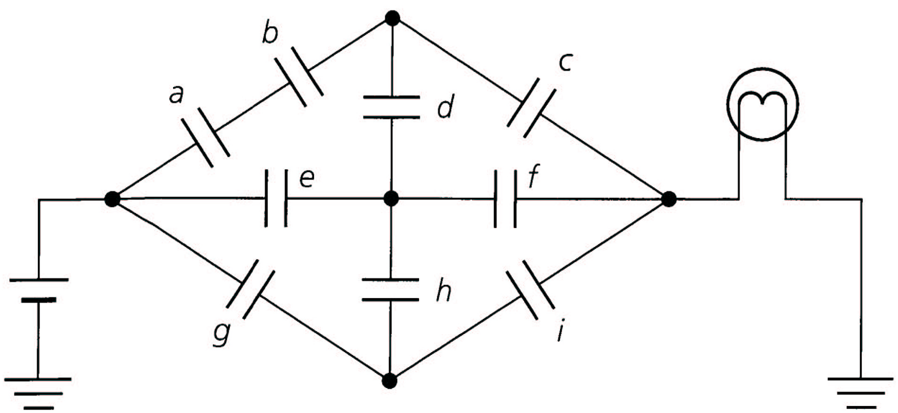


Figure 11.65

- Switches maybe by default open or closed

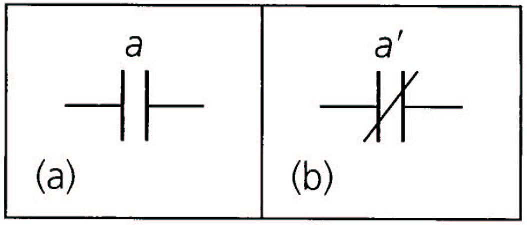
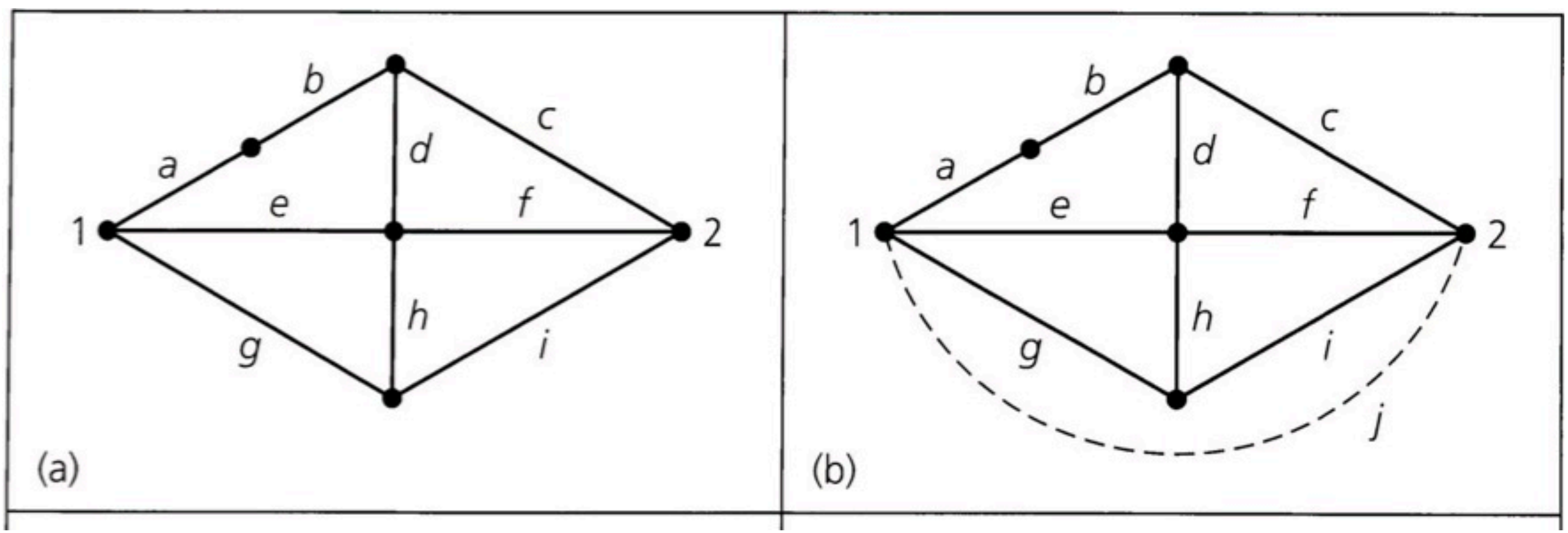


Figure 11.66

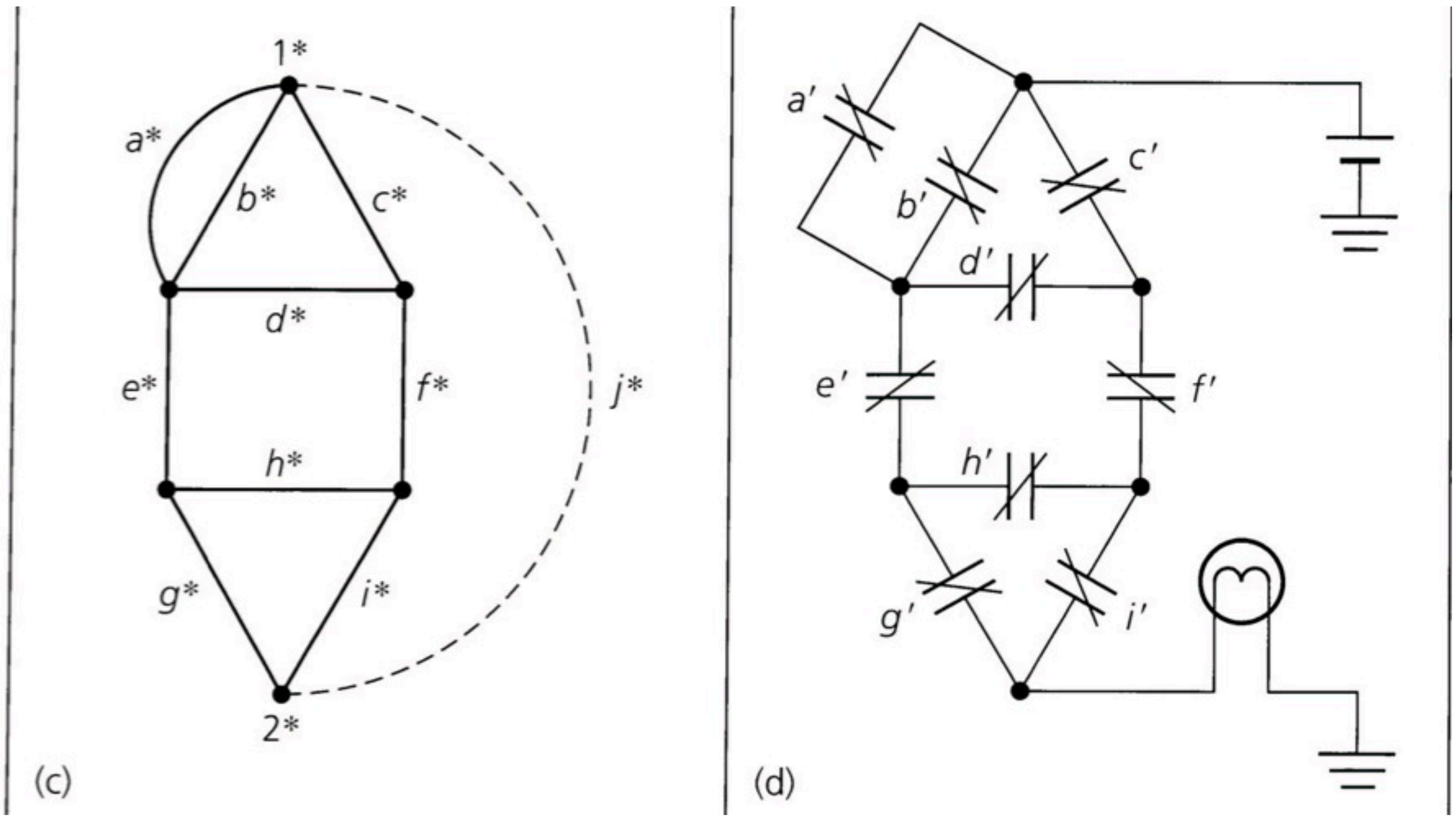
A Real Application (cont.)

- One-terminal-pair graph: two terminal vertices
- Planar-one-terminal-pair graph: if adding an edge connecting the terminals still leads to a planar graph



A Real Application (cont.)

- The dual graph!



A Real Application (cont.)

- What happens if we close a,b,c,j in G? How do they affect G^d?
- How about closing c,d,e,g,j in G?
- Cycles versus disconnections!

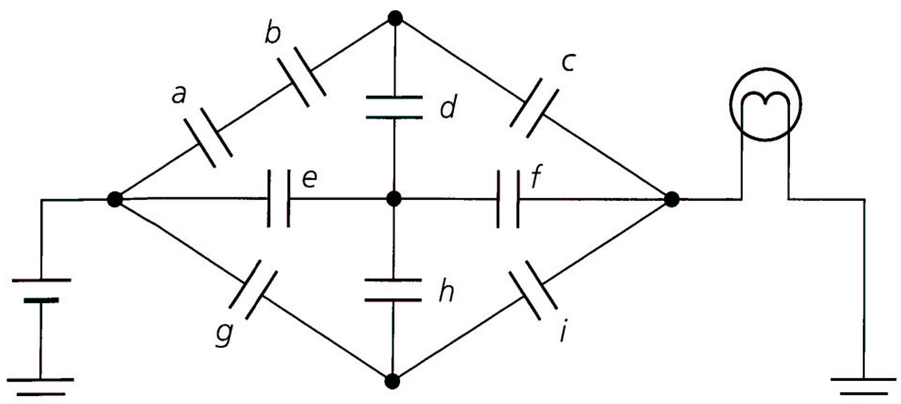
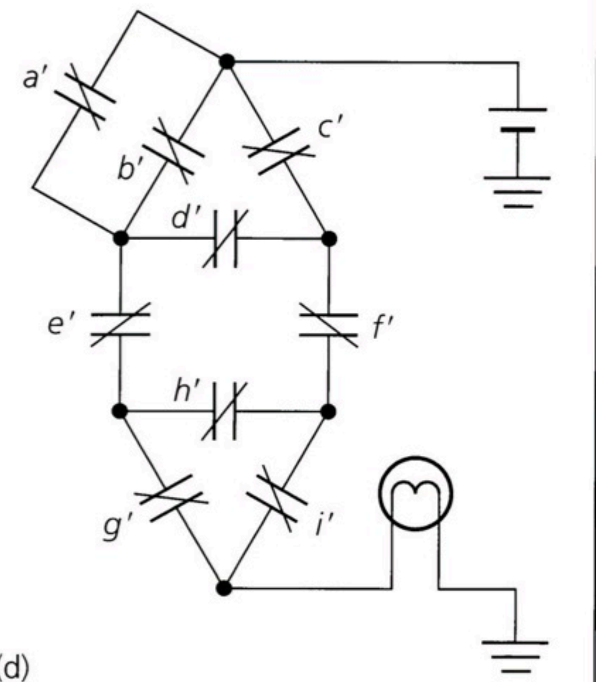


Figure 11.65



(d)

Outline

11.1 Definitions and Examples

11.2 Subgraphs, Complements, and Graph Isomorphism

11.3 Vertex Degree: Euler Trails and Circuits

11.4 Planar Graphs

11.5 Hamilton Paths and Cycles

11.6 Graph Coloring and Chromatic Polynomials

Hamilton Cycle

- G=(V,E) is a graph or multigraph with |V|>=3. We say G has a Hamilton cycle if there is a cycle in G containing every vertex of V.
- A Hamilton path is a path in G that contains every vertex.

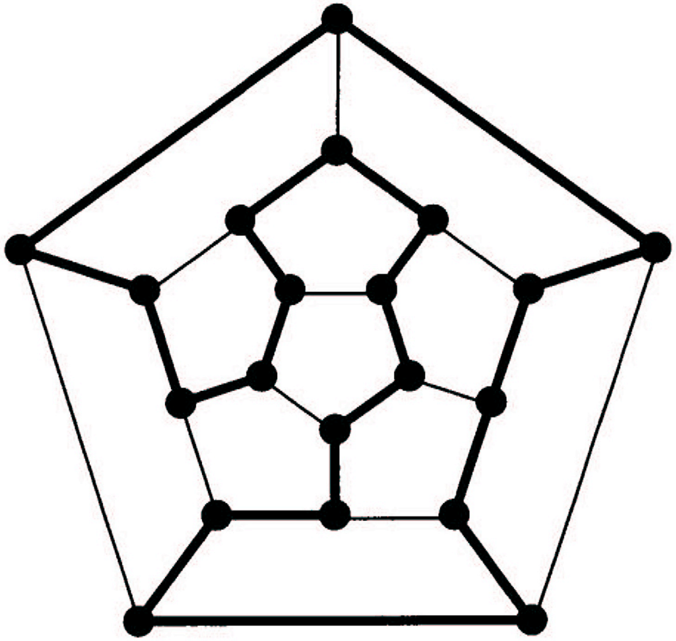


Figure 11.77

Hamilton Cycle (cont.)

- Sounds like Euler circuits (trails): edges for Euler circuits (trails)
- Unlike Euler circuits (trails), we don't know any necessary and sufficient conditions for the existence of Hamilton cycle.
  - Theorems are for necessary or sufficient conditions
- Ex 11.26: Find Hamilton cycles in Q_2 and Q_3.

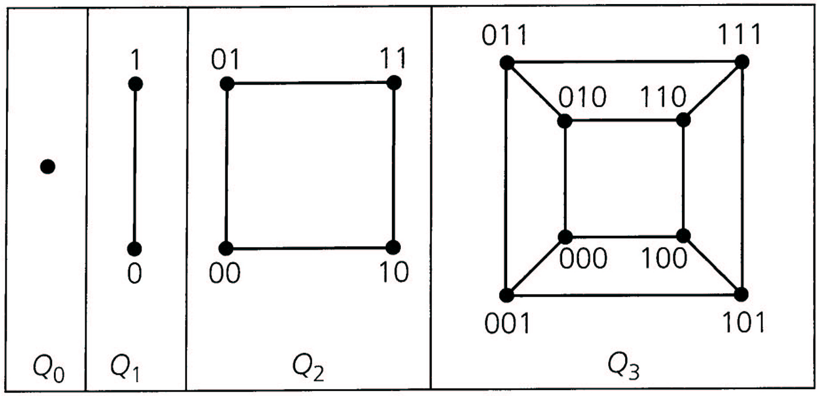


Figure 11.35

Another Example

- Ex 11.27: Find a Hamilton cycle in the figure.
- Is there a Hamilton cycle in this graph?

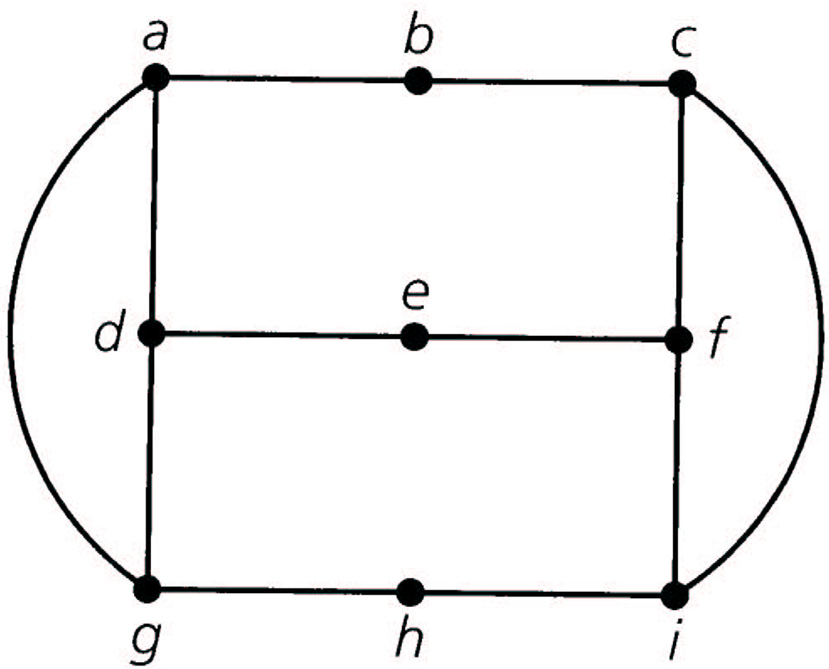


Figure 11.78

Finding Hamilton Cycle

- If G has a Hamilton cycle, then deg(v)>=2 for all vertices
- If a is *a* vertex and deg(*a*)=*2*, then the two edges incident with *a* must appear in every Hamilton cycle
- If a is a vertex and deg(a)>2, then once we pass through a, all other edges incident with a are removed (as they cannot be part of the cycle)
- We cannot find a Hamilton cycle for a subgraph of G unless it contains all vertices

An Example

- Ex 11.28: We re-label Subfig (a) into (b) by alternating x and y.
- Should contain alternating x's (5 of them) and y's (5 of them). But we don't have enough x, so Subfig (a) doesn't contain a Hamilton cycle

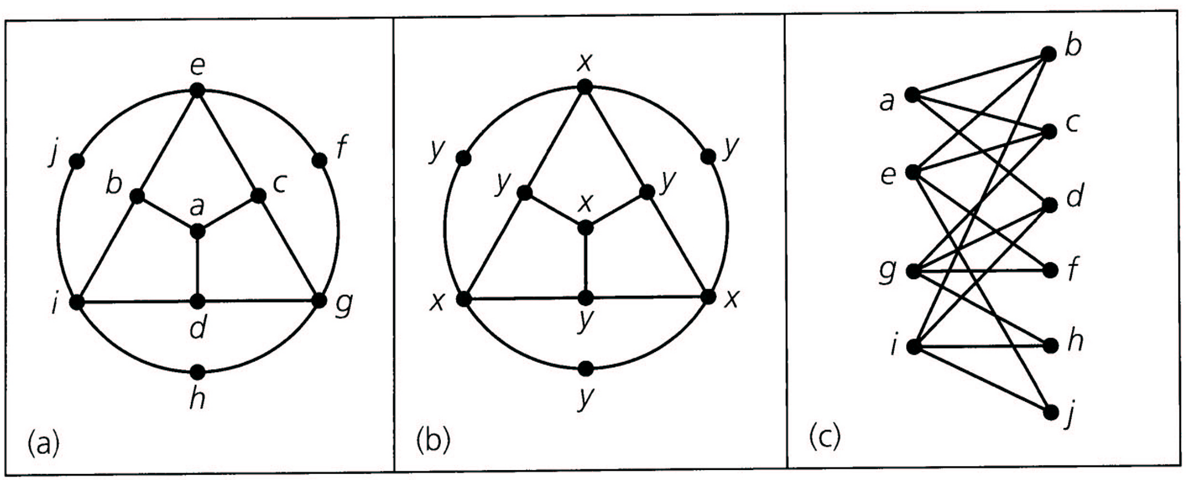


Figure 11.79

In a Complete Graph

- Ex 11.29: In a science camp, 17 students have lunch together daily at a circular table. Students sit next to two different colleagues every day. For how many days can they do this? Any systematic way to arrange the seats?
- Each arrangement is in fact a Hamilton cycle of all students.
- There are n vertices and n(n-1)/2 edges. It is a K_n.
- Since every cycle has n edges, we have at most (n-1)/2 cycles ← these many days

In a Complete Graph (cont.)

- Systematic approach: rotate the Hamilton cycle by $\frac{1}{n-1}2\pi$
- For (n-1)/2 times

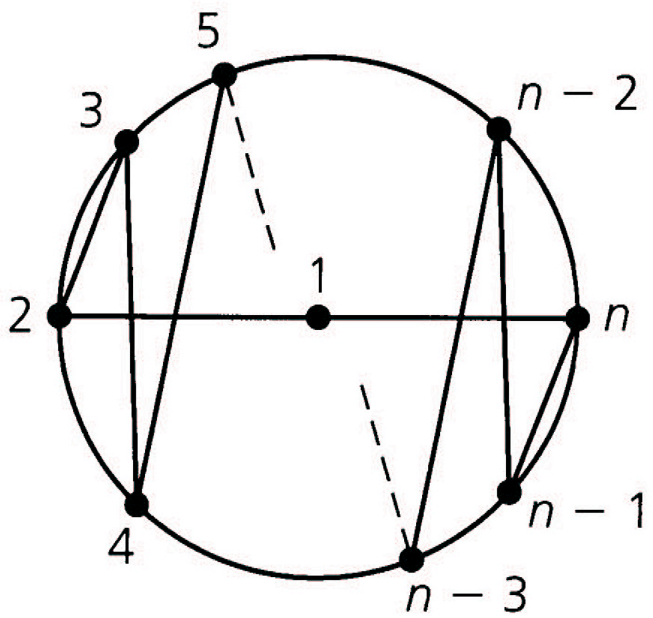


Figure 11.80

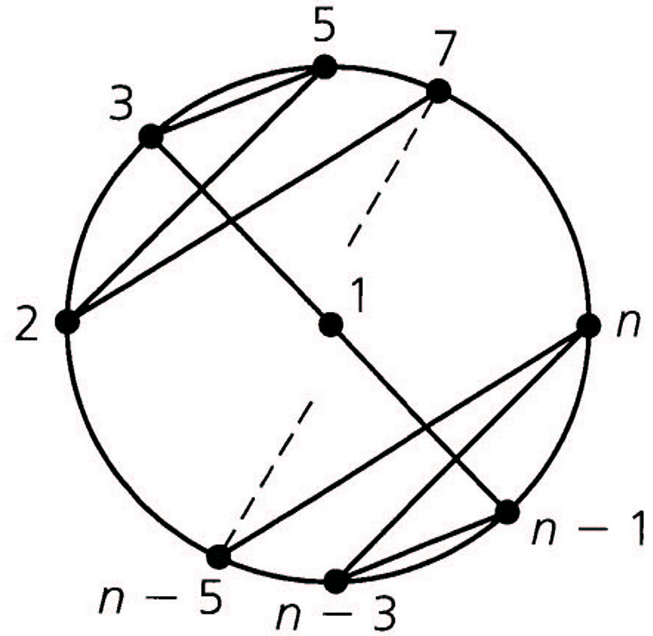


Figure 11.81

Due to L. Redei

- Let K_n^* be a complete directed graph. That is it has n vertices and for distinct vertices x,y, exactly one of (x,y) or (x,y) exists in K_n^* .It is called a tournament. Such a graph always contains a directed Hamilton path.
- Proof Sketch: By incrementally adding edges to a path until it convers all vertices. See text for details.
- Ex 11.30: In a round-robin tournament each player plays every other player once. A Hamilton path over the directed graph lists the players so that each has beaten the next one on the list.

Sufficient Condition for Hamilton Path

- G=(V,E) is a loop-free graph with |V|=n>=2. If deg(x)+deg(y)>=n-1 for all distinct vertices x, y, then G has a Hamilton path.
- Proof: Given in the text.
- Corollary: If deg(v)>=(n-1)/2 for all vertex v, the graph has a Hamilton path.

Sufficient Condition for Hamilton Cycle

- G=(V,E) is a loop-free graph with |V|=n>=3. If deg(x)+deg(y)>=n for all nonadjacent vertices x,y, then G contains a Hamilton cycle.
- Proof: Given in the text.

- Corollary 11.5: If deg(v)>=n/2 for all ertex v, then G has a Hamilton cycle.
- Corollary 11.6: If $|E| \geqslant \binom{n-1}{2} + 2$, then G has a Hamilton cycle

Traveling Salesman Problem

- A traveling salesperson leaves his/her home, and must visit a set of locations before returning. The goal is to find the order to visit the locations, to maximize his/her efficiency.
  - Objectives can be cost or distance.
- TSP can be modeled as a labeled graph, where the most efficient Hamilton cycle is wanted.

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Proper Coloring

- For an undirected graph G=(V,E), if a coloring scheme of G is called a proper coloring if any adjacent vertices have different colors.
- The minimum number of colors needed to properly color G is called the chromatic number of G, $\chi(G)$
- Example: An aquarium has 10 kinds of fish. Certain types of fish cannot be put into the same fish tanks. What is the minimum number of tanks we need to hold all of them?

Four Color Theorem

- Four colors are enough to color any planar maps
  - The first computer-assisted proof in 1976
  - 1936 subgraphs
- For nonplanar graphs, more than 4 colors might be needed!

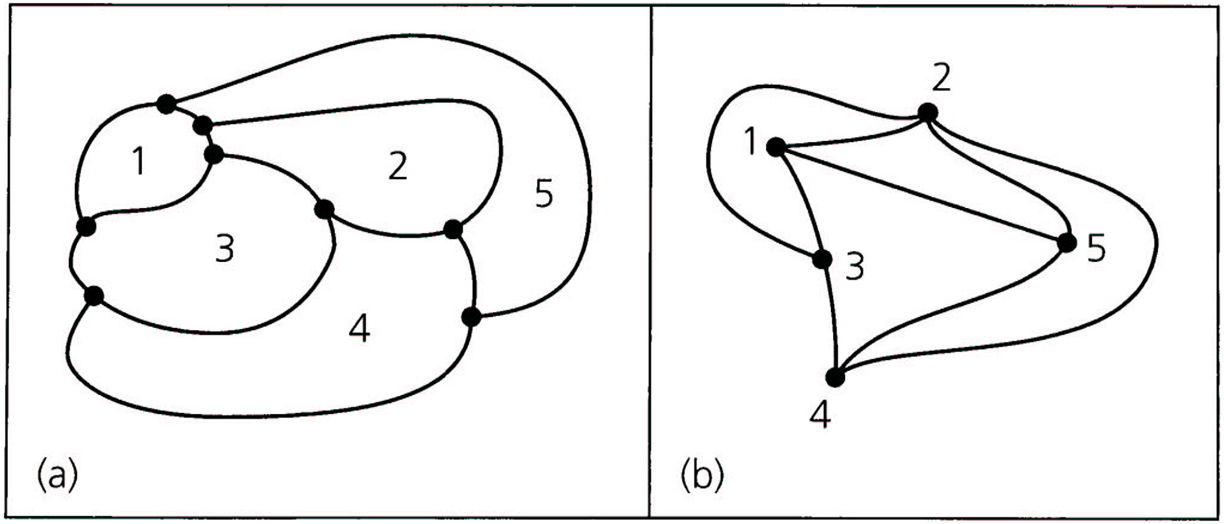

Figure 11.64

Very Simple Example

- Ex 11.31: What is $\chi(G)$?
  - Show $\chi(G) \geqslant 3$ by showing K_3 is a subgraph of G
  - Show $\chi(G) \leqslant 3$ by labeling the vertices (showing a sample coloring scheme.

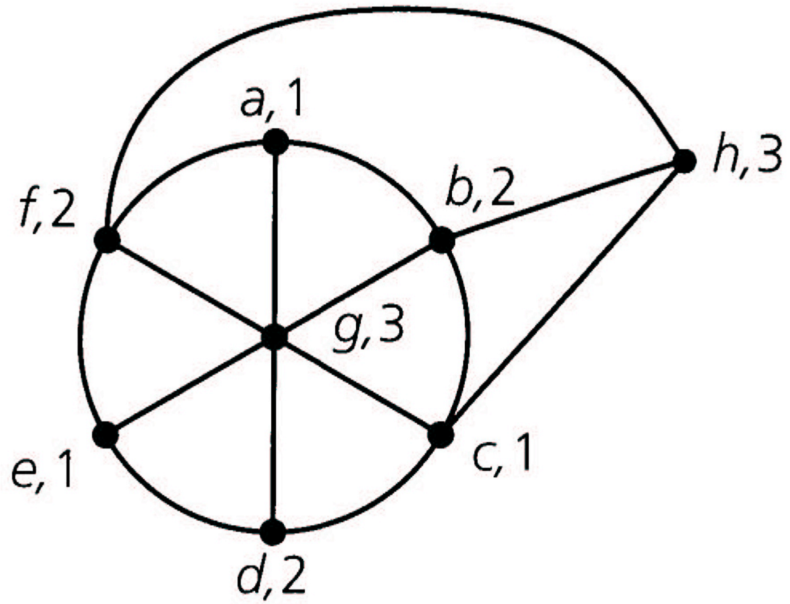

Figure 11.87

More Examples

- Ex 11.32, 11.33:
  - $\chi(K_n) = n$
  - Show $\chi(G) = 4$

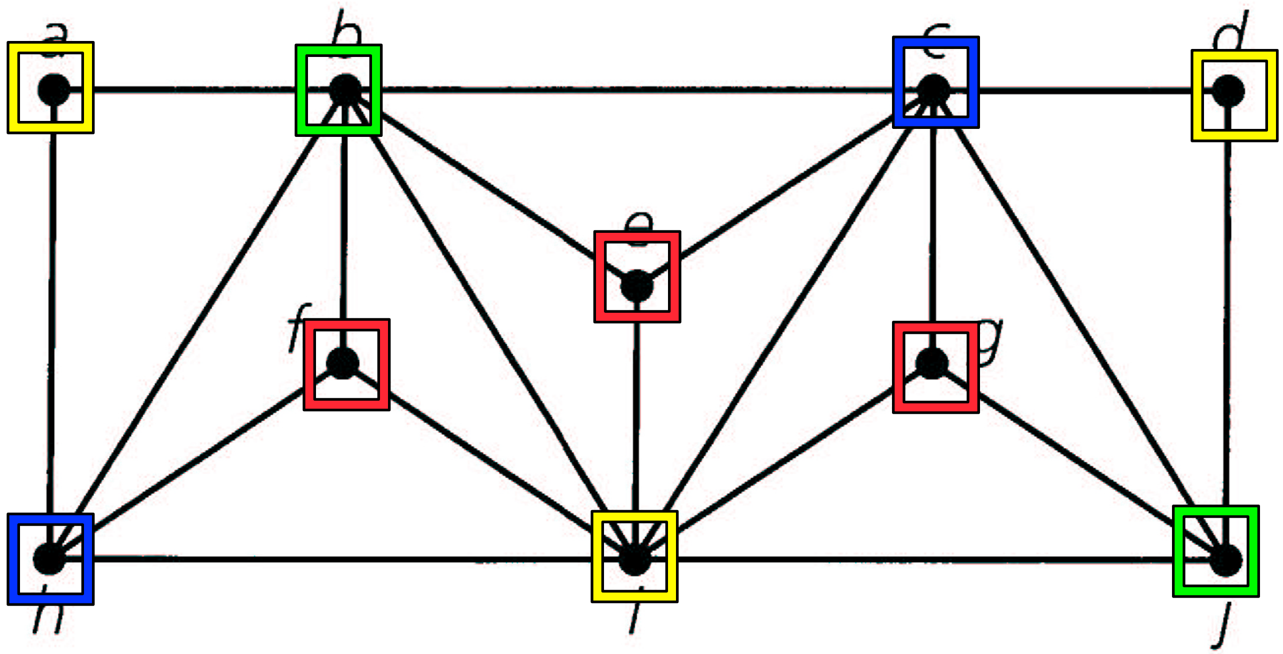

Figure 11.88

Chromatic Polynomial

- Let λ be the number of available number. The chromatic polynomial $P(G, \lambda)$ tells us how many different ways we can properly color G
- A proper coloring scheme is a function from domain V to codomain $\{1, 2, 3, \ldots, \lambda\}$
- Different ways mean different functions

Simple Examples

- Ex 11.34:
  - For graph G with n vertices and zero edge, $P(G, \lambda) = \lambda^n$
  - For K_n , $P(G, \lambda) = \lambda(\lambda-1)(\lambda-2)\cdots(\lambda-n+1)$, $\lambda \geqslant n$
    - $P(G, \lambda) = 0, \text{ o.w.}$
  - If G consists of components, its P fcn is the product of all components' P fcns.

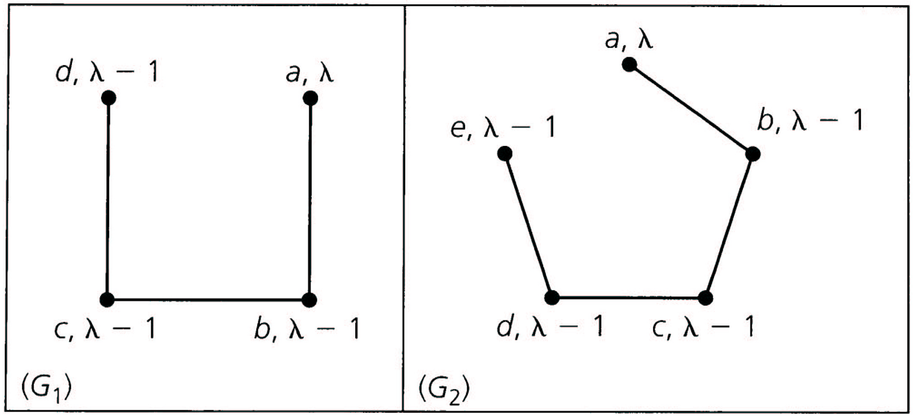

Figure 11.89

Two Special Subgraphs

- What happen if G is a connected (and a rather large) graph?
- For e={a,b} is an edge of G
  - let subgraph G_e=G-e
  - coalescing {a,b} into a subgraph G'_e
- Ex 11.35:

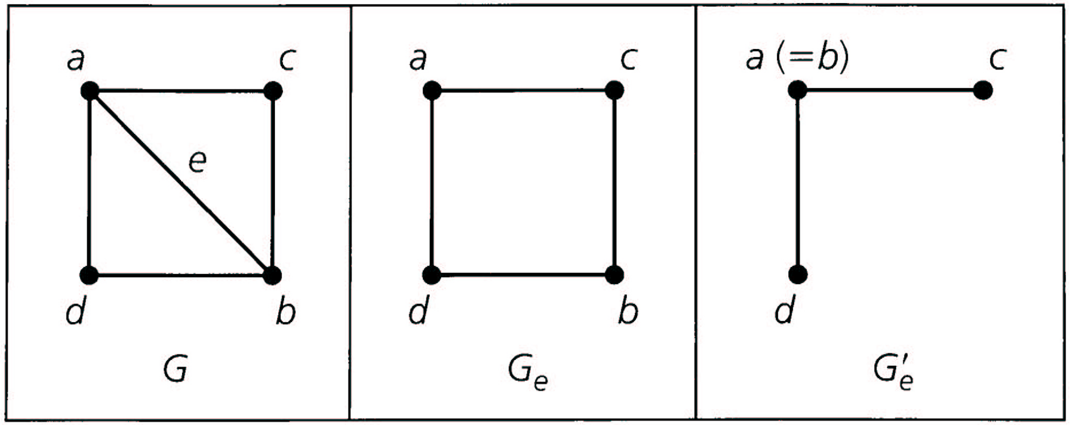

Figure 11.90

$$P(G_e, \lambda) = P(G, \lambda) + P(G'_e, \lambda)$$

Example

- Ex 11.36: How to get $P(G,\lambda) = \lambda^4 - 4\lambda^3 + 6\lambda^2 - 3\lambda$?

$P(G,\lambda)$ $P(G_e,\lambda)$ $P(G'_e,\lambda)$

- How to determine $\chi(G)$?

Properties of Chromatic Poly.

- For any graph G, the constant term of $P(G,\lambda)$ is 0
  - Otherwise P(G,0)>0, means we can color the graph using zero color!
- For graph G with |E| >0. The sum of the coefficient of $P(G,\lambda)$ is 0
  - At least an edge, need at least two colors, so P(G,1)=0

Another Approach

- For a connected graph, we can also add more edges until we get a complete graph, in order to compute its chromatic polynomial

- For two vertices a,b where {a,b} is not an edge in G. Let G_e^+ be the graph G with an addition edge {a,b}, coalescing a,b in G gives a subgraph G_e^{++}. We have $P(G_e^+, \lambda) = P(G, \lambda) + P(G_e^{++}, \lambda)$

Complete Graphs

- G contains two subgraphs G_1, G_2. If $G = G_1 \cup G_2$ and $G = G_1 \cap G_2 = K_n$
- Then, $P(G,\lambda) = \dfrac{P(G_1,\lambda)P(G_2,\lambda)}{\lambda^{(n)}}$
- Proof Sketch: There are $\lambda^{(n)}$ ways to color K_n. There are $\dfrac{P(G_1,\lambda)}{\lambda^{(n)}}$ ways to color the rest vertices of G_1 and $\dfrac{P(G_2,\lambda)}{\lambda^{(n)}}$ ways to color the rest vertices of G_2.
- Then, $P(G,\lambda) = P(K_n,\lambda)\dfrac{P(G_1,\lambda)}{\lambda^{(n)}}\dfrac{P(G_2,\lambda)}{\lambda^{(n)}}$
- Ex 11.39: Left as exercise

Take-home Exercises

- Exercise 11.1: 2, 3, 5, 8, 13
- Exercise 11.2: 1, 2, 3, 9, 15
- Exercise 11.3: 3, 4, 5, 20, 23
- Exercise 11.4: 2, 3, 13, 14, 26
- Exercise 11.5: 1, 3, 4, 6, 19
- Exercise 11.6: 1, 2, 5, 6, 13