SOLUTION

Ex 9.1: 1, 3, 4

Ex 9.2: 1, 2, 9, 18, 19

Ex 9.3: 1, 3, 4, 6

Ex 9.4: 1, 2, 5, 6, 9

Ex 9.5: 1, 2, 3, 5

Ex 9.1: (1)

- The number of integer solutions for the given equations is the coefficient of

a) x^{20} in $(1+x+x^2+\cdots+x^7)^4$.

b) x^{20} in $(1+x+x^2+\cdots+x^{20})^2(1+x^2+x^4+\cdots+x^{20})^2$ or $(1+x+x^2+\cdots)^2(1+x^2+x^4+\cdots)^2$.

c) x^{30} in $(x^2+x^3+x^4)(x^3+x^4+\cdots+x^8)^4$.

d) x^{30} in $(1+x+x^2+\cdots+x^{30})^3$ $(1+x^2+x^4+\cdots+x^{30})(x+x^3+x^5+\cdots+x^{29})$ or $(1+x+x^2+\cdots)^3(1+x^2+x^4+\cdots)(x+x^3+x^5+\cdots)$.

Ex 9.1: (3)

a) The generating function is either $(1 + x + x^2 + \cdots + x^{10})^6$or $(1 + x + x^2 + \cdots)^6$.
[The number of ways to select 10 candy bars is the coefficient of x^{10} in either case.]

b) The generating function is either $(1 + x + x^2 + \cdots + x^r)^n$or $(1 + x + x^2 + \cdots)^n$.
[The number of selections of r objects is the coefficient of x^r in either case.]

Ex 9.1: (4)

a) The first factor counts the pennies; the nickels are counted by the second factor.

b) $f(x) = (1 + x + x^2 + \cdots)(1 + x^5 + x^{10} + \cdots)(1 + x^{10} + x^{20} + \cdots)$.

Ex 9.2: (1)

a) $(1+x)^8$

b) $8(1+x)^7$

c) $(1+x)^{-1}$

d) $\frac{6x^3}{1+x}$

e) $(1-x^2)^{-1}$

f) $\frac{x^2}{1-ax}$

Ex 9.2: (2)

a) $-27,54,-36,8,0,0,0,\ldots$

b) $0,0,0,1,1,1,1,1,\ldots$

c) $f(x)=\frac{x^3}{1-x^2}=x^3[1+x^2+x^4+\cdots]=x^3+x^5+x^7+\cdots$, so $f(x)$ generates the sequence $0,0,0,1,0,1,0,1,0,1,\ldots$

d) $f(x)=\frac{1}{1+3x}=1+(-3x)+(-3x)^2+(-3x)^3+\cdots$, so $f(x)$ generates the sequence $1,-3,3^2,(-3)^3,\ldots$

e) $f(x)=\frac{1}{3-x}=\left(\frac{1}{3}\right)\left[\frac{1}{1-\frac{x}{3}}\right]=\left(\frac{1}{3}\right)\left[1+\frac{x}{3}+\left(\frac{x}{3}\right)^2+\cdots\right]$, so $f(x)$ generates the sequence 1/3, (1/3)^2, (1/3)^3, ...

f) $f(x)=\frac{1}{1-x}+3x^7-11=(1+x+x^2+x^3+\cdots)+3x^7-11$, so $f(x)$ generates the sequence $a_0,a_1,a_2,\ldots$, where $a_0=-10$, $a_7=4$, and $a_i=1$ for all $i\neq 0,7$

Ex 9.2: (9)

a) 0

b) $\binom{-3}{12}(-1)^{12} - 5\binom{-3}{14}(-1)^{14} = \binom{14}{12} - 5\binom{16}{14}$

c) $\binom{-4}{15}(-1)^{15} + \binom{4}{1}\binom{-4}{14}(-1)^{14} + \binom{4}{2}\binom{-4}{13}(-1)^{13} + \binom{4}{3}\binom{-4}{12}(-1)^{12} + \binom{4}{4}\binom{-4}{11}(-1)^{11} = \binom{18}{15} + \binom{4}{1}\binom{17}{14} + \binom{4}{2}\binom{16}{13} + \binom{4}{3}\binom{15}{12} + \binom{14}{11}$

Ex 9.2: (18)

- $(1-4x)^{-\frac{1}{2}} = \left[\binom{-\frac{1}{2}}{0} + \binom{-\frac{1}{2}}{1}(-4x) + \binom{-\frac{1}{2}}{2}(-4x)^2 + \cdots\right]$. The coefficient of x^n is $\binom{-\frac{1}{2}}{n}(-4)^n = \frac{\left[\left(\left(-\frac{1}{2}\right)-n+1\right)\left(\left(-\frac{1}{2}\right)-n+2\right)\cdots\left(\left(-\frac{1}{2}\right)-1\right)\left(-\frac{1}{2}\right)\right]}{n!}(-4)^n = \frac{[(1+2n-2)(1+2n-4)\cdots(1+2)(1)]}{n!}(2)^n = \frac{[(2n-1)(2n-3)\cdots(5)(3)(1)]}{n!}(2)^n = \frac{[(2n-1)(2n-3)\cdots(5)(3)(1)](2^n)(n!)}{n!n!} = \frac{(2n)!}{n!n!} = \binom{2n}{n}$

Ex 9.2: (19)

a) There are $2^{8-1} = 2^7$compositions of 8 and $2^{\left\lfloor \frac{8}{2} \right\rfloor} = 2^4$ palindromes of 8. Assuming each composition of 8 has the same probability of being generated, the probability a palindrome of 8 is generated is $\frac{2^4}{2^7} = \frac{1}{8}$.

b) Assuming each composition of n has the same probability of being generated, the probability a palindrome of n is generated is $\frac{2^{\left\lfloor \frac{n}{2} \right\rfloor}}{2^{n-1}} = 2^{\left\lfloor \frac{n}{2} \right\rfloor - n + 1} = 2^{1 - \left\lceil \frac{n}{2} \right\rceil}$.

Ex 9.3: (1)

- 7;
  6 + 1;
  5 + 2;
  5 + 1 + 1;
  4 + 3;
  4 + 2 + 1;
  4 + 1 + 1 + 1;
  3 + 3 + 1;
  3 + 2 + 2;
  3 + 2 + 1 + 1;
  3 + 1 + 1 + 1 + 1;
  2 + 2 + 2 + 1;
  2 + 2 + 1 + 1 + 1;
  2 + 1 + 1 + 1 + 1 + 1;
  1 + 1 + 1 + 1 + 1 + 1 + 1

Ex 9.3: (3)

- The number of partitions of 6 into 1's, 2's, and 3's is 7.

Ex 9.3: (4)

a) $\left[\frac{1}{1-t^2}\right]\left[\frac{1}{1-t^3}\right]\left[\frac{1}{1-t^5}\right]\left[\frac{1}{1-t^7}\right]$

b) $\left[\frac{1}{1-t^2}\right]\left[\frac{t^{12}}{1-t^3}\right]\left[\frac{t^{20}}{1-t^5}\right]\left[\frac{t^{35}}{1-t^7}\right]$

Ex 9.3: (6)

a) $f(x) = (1 + x + x^2 + \cdots + x^5)(1 + x^2 + x^4 + \cdots + x^{10}) \ldots$
$= \prod_{i=1}^{\infty}\left(1 + x^i + x^{2i} + \cdots + x^{5i}\right) = \prod_{i=1}^{\infty} \frac{1-x^{6i}}{1-x^i}$

b) $\prod_{i=1}^{12}\left(1 + x^i + x^{2i} + \cdots + x^{5i}\right) = \prod_{i=1}^{\infty} \frac{1-x^{6i}}{1-x^i}$

Ex 9.4: (1)

a) e^{-x}

b) e^{2x}

c) e^{-ax}

d) e^{a^2x}

e) ae^{a^2x}

f) xe^{2x}

Ex 9.4: (2)

a) $f(x) = 3e^{3x} = 3\sum_{i=0}^{\infty}\frac{(3x)^i}{i!}$, so $f(x)$ is the exponential generating function for the sequence 3,3^2,3^3, ...

b) $f(x) = 6e^{5x} - 3e^{2x} = 6\sum_{i=0}^{\infty}\frac{(5x)^i}{i!} - 3\sum_{i=0}^{\infty}\frac{(2x)^i}{i!}$, so $f(x)$ is the exponential generating function for the sequence $3, 24, 138, \ldots, 6(5^n) - 3(2^n), \ldots$

c) $1,1,3,1,1,1,1, \ldots$

d) $1,9,14,-10,2^4,2^5,2^6, \ldots$

e) $f(x) = 1 + x + x^2 + \cdots = \sum_{i=0}^{\infty} i!\left(\frac{x^i}{i!}\right)$, so $f(x)$ is the exponential generating function for the sequence 0!, 1!, 2!, 3!, ...

f) $f(x) = 3[1 + 2x + (2x)^2 + \cdots] + \sum_{i=0}^{\infty}\frac{x^i}{i!}$, so $f(x)$is the exponential generating function for the sequence $4,7,25,145, \ldots, (3n!)2^n + 1, \ldots$

Ex 9.4: (5)

- We find that $\frac{1}{1-x} = 1 + x + x^2 + \cdots = 0!\frac{x^0}{0!} + 1!\frac{x^1}{1!} + 2!\frac{x^2}{2!} + \cdots$, so $\frac{1}{1-x}$ is the exponential generating function for the sequence 0!, 1!, 2!, ...

Ex 9.4: (6)

a) (i) $(1+x)^2\left(1+x+\left(\frac{x^2}{2!}\right)\right)^2$

(ii) $(1+x)\left(1+x+\left(\frac{x^2}{2!}\right)\right)\left(1+x+\left(\frac{x^2}{2!}\right)+\left(\frac{x^3}{3!}\right)+\left(\frac{x^4}{4!}\right)\right)^2$

(iii) $(1+x)^3\left(1+x+\left(\frac{x^2}{2!}\right)\right)^4$

b) $(1+x)\left(1+x+\left(\frac{x^2}{2!}\right)\right)\left(1+x+\left(\frac{x^2}{2!}\right)+\left(\frac{x^3}{3!}\right)+\left(\frac{x^4}{4!}\right)\right)$
$\left(\left(\frac{x^2}{2!}\right)+\left(\frac{x^3}{3!}\right)+\left(\frac{x^4}{4!}\right)\right)$

Ex 9.4: (9)

a) $\frac{1}{2}\frac{\left[3^{20}+1\right]}{3^{20}}$

b) $\frac{1}{4}\frac{\left[3^{20}+3\right]}{3^{20}}$

c) $\frac{1}{2}\frac{\left[3^{20}-1\right]}{3^{20}}$

d) $\frac{1}{2}\frac{\left[3^{20}-1\right]}{3^{20}}$

e) $\frac{1}{2}\frac{\left[3^{20}+1\right]}{3^{20}}$

Ex 9.5: (1)

a) $1+x+x^2$ is the generating function for the sequence 1,1,1,0,0,0, ..., so $\frac{1+x+x^2}{1-x}$ is the generating function for the sequence $1,1+1,1+1+1,1+1+1+0, \ldots$ – that is the sequence 1,2,3,3, ...

b) $1+x+x^2+x^3$ is the generating function for the sequence 1,1,1,1,0,0,0, ..., so $\frac{1+x+x^2+x^3}{1-x}$ is the generating function for the sequence $1,1+1,1+1+1,1+1+1+1,1+1+1+1+0,1+1+1+1+0+0, \ldots$ – that is, the sequence 1,2,3,4,4,4, ...

c) $1+2x$ is the generating function for the sequence 1,2,0,0,0,0, ..., so $\frac{1+2x}{1-x}$ is the generating function for the sequence $1,1+2,1+2+0,1+2+0+0, \ldots$ – that is , the sequence 1,3,3,3, ... Consequently, $\frac{1}{1-\mathrm{x}}\left[\frac{1+2\mathrm{x}}{1-\mathrm{x}}\right]=\frac{1+2x}{(1-x)^2}$ is the generating function for the sequence $1,1+3,1+3+3,1+3+3+3+3, \ldots$ – that is, the sequence 1,4,7,10, ...

Ex 9.5: (2)

a) (i) x

(ii) $\frac{x}{1-x}$

(iii) $\frac{x}{(1-x)^2}$

(iv) $\frac{x}{(1-x)^3}$

b) $\sum_{k=1}^{n} k =$ the coefficient of x^n in $\frac{x}{(1-x)^3}$

$=$ the coefficient of x^n in $x(1-x)^{-3}$

$=$ the coefficient of x^{n-1} in $(1-x)^{-3}$

$= \binom{-3}{n-1}(-1)^{n-1} = (-1)^{n-1}\binom{3+(n-1)-1}{n-1}(-1)^{n-1}$

$= \binom{n+1}{n-1} = \frac{1}{2}(n+1)n$

Ex 9.5: (3)

- $f(x) = \frac{[x(1+x)]}{(1-x)^3}$ generates $0^2, 1^2, 2^2, \ldots$;
  $\frac{[x(1+x)]}{(1-x)^3} = 0^2 + 1^2 x + 2^2 x^2 + \cdots$;
  $\left(\frac{d}{dx}\right)\left[\frac{x+x^2}{(1-x)^3}\right] = 1^3 + 2^3 x + 3^3 x^2 + \cdots$;
  $x\left(\frac{d}{dx}\right)\left[\frac{x+x^2}{(1-x)^3}\right] = 0^3 + 1^3 x + 2^3 x^2 + 3^3 x^3 + \cdots$;
  $\left(\frac{d}{dx}\right)\left[\frac{x+x^2}{(1-x)^3}\right] = \frac{x^2+4x+1}{(1-x)^4}$, so $\frac{x(x^2+4x+1)}{(1-x)^5}$ generates $0^3, 0^3 + 1^3, 0^3 + 1^3 + 2^3, \ldots$, and the coefficient of x^n is $\sum_{i=0}^{n} i^3$.
- $(x^3 + 4x^2 + x)(1-x)^{-5} = (x^3 + 4x^2 + x)\left[\binom{-5}{0} + \binom{-5}{1}(-x) + \binom{-5}{2}(-x)^2 + \cdots\right]$. Here the coefficient of x^n is $\binom{-5}{n-3}(-1)^{n-3} + 4\binom{-5}{n-2}(-1)^{n-2} + \binom{-5}{n-1}(-1)^{n-1}$
  $= \binom{n+1}{n-3} + 4\binom{n+2}{n-2} + \binom{n+3}{n-1}$
  $= \left(\frac{1}{4!}\right)[(n+1)n(n-1)(n-2) + 4(n+2)(n+1)n(n-1) + (n+3)(n+2)(n+1)n]$

$$= \left[\frac{(n+1)n}{4!}\right](6n^2 + 6n) = \frac{(n+1)n(n^2+n)}{4} = \left[\frac{(n+1)n}{2}\right]^2$$

Ex 9.5: (5)

- $(1-x)f(x) = (1-x)(a_0 + a_1x + a_2x^2 + \cdots) = a_0 + (a_1 - a_0)x + (a_2 - a_1)x^2 + \cdots$, so $(1-x)f(x)$ is the generating function for the sequence $a_0, a_1 - a_0, a_2 - a_1, a_3 - a_2, \ldots$.