Name: Student ID:

Quiz #9 5%

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This is a closed book test. Any academic dishonesty will automatically lead to zero point.

1) (1%) Find the number of generating function for the number of integer solutions to the equation: $c_1 + c2 + c3 + c4 = 20$, where $-3 \leq c_1$, $-3 \leq c_2$, $-5 \leq c_3 \leq 5$, and $0 \leq c_4$

   Solution:

   $(3 + c_1) + (3 + c_2) + (5 + c_3) + c_4 = 31$

   $\Rightarrow x_1 + x_2 + x_3 + x_4 = 31$, $0 \leq x_1, x_2, x_3$, $0 \leq x_4 \leq 10$.

   Consequently, the answer is the coefficient of x^{31} in the generating function:

   $(1 + x + x^2 + \cdots)^3(1 + x + x^2 + \cdots + x^{10})$.

2) (1%) In how many ways can Tracy select n marbles from a large bag of blue, red, and yellow marbles (all of the same size) if the selection must include an even number of blue ones ?

   Solution:

   We need the coefficient of x^n in $(1+x+x^2+x^3\cdots)^2(1+x^2+x^4+\cdots) = (\frac{1}{1-x})^2(\frac{1}{1-x^2}) = (\frac{1}{1-x})^3(\frac{1}{1+x})$.Using a partial fraction decomposition, $(\frac{1}{1-x})^3(\frac{1}{1+x}) = \frac{1/8}{1+x} + \frac{1/8}{1-x} + \frac{1/4}{(1-x)^2} + \frac{1/2}{(1-x)^3}$, where the coefficient of x^n is $(-1)^n\frac{1}{8} + \frac{1}{8} + \frac{1}{4}\binom{-2}{n}(-1)^n + \frac{1}{2}\binom{-3}{n}(-1)^n$.

3) (2%) Find the generating function for the number of integer solutions of: (a) $2w + 3x + 5y + 7z = n$, $0 \leq w, x, y, z$ and (b) $2w + 3x + 5y + 7z = n$, $0 \leq w$, $4 \leq x, y$, $5 \leq z$.

   Solution:

   a) $\frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^5} \cdot \frac{1}{1-x^7}$
   b) $\frac{1}{1-x^2} \cdot \frac{x^{12}}{1-x^3} \cdot \frac{x^{20}}{1-x^5} \cdot \frac{x^{35}}{1-x^7}$

4) (1%) In each of the following, the function $f(x)$ is the exponential generating function for the sequence $a_0, a_1, a_2 \cdots$, whereas the function $g(x)$ is the exponential generating function for the sequence $b_0, b_1, b_2 \cdots$. Express $g(x)$ in terms of $f(x)$ if

a) $b_3 = 3$

   $b_n = a_n$, where $n \in \mathbf{N}, n \neq 3$

b) $a_n = 5^n$, where $n \in \mathbf{N}$

   $b_3 = -1$

   $b_n = a_n$, where $n \in \mathbf{N}, n \neq 3$

Solution:

a) $g(x) = f(x) + (3 - a_3)(x^3/3!)$

b) $g(x) = f(x) + (-1 - a_3)(x^3/3!)$