Name: Student ID:

Quiz #5 5%

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This is a closed book test. Any academic dishonesty will automatically lead to zero point.

1) (1%) $A = \{1, 2, 3, 4, 5\}$ and $B = \{w, x, y, z\}$. How many elements are in $\mathscr{P}(A \times B)$, the power set of $A \times B$? If $|A| = m$ and $|B| = n$, how many elements are in $\mathscr{P}(A \times B)$?

   Solution:

   a) Since $|A| = 5$ and $|B| = 4$, we have $|A \times B| = |A|\,|B| = 5 \cdot 4 = 20$. Hence, $|\mathscr{P}(A \times B)| = 2^{20}$
   b) Similarly, $|A \times B| = |A|\,|B| = m \cdot n = mn$. Hence, $|\mathscr{P}(A \times B)| = 2^{mn}$

2) (1%) Let $A = \{1, 2, 3, 4, 5\}$, $B = \{w, x, y, z\}$, $A_1 = \{2, 3, 5\} \subseteq A$, and $g : A_1 \to B$. In how many ways can g be extended to a function $f : A \to B$?

   Solution: The extension must include $f(1)$ and $f(4)$. There are four choices for each of 1 and 4, since $|B| = 4$. Hence, there are 4^2 ways to extend the given function.

3) (1.5%)

   a) Verify that $5^7 = \sum_{i=1}^{5} \binom{5}{i}(i!)S(7, i)$
   b) Provide a combinatorial argument to justify for all $m, n \in \mathbb{Z}^+$, $m^n = \sum_{i=1}^{m} \binom{m}{i}(i!)S(n, i)$

   Note that $S(m, n) = \frac{1}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{n-k}(n-k)^m$. It denotes the number of ways to distribute m distinct objects into n nonempty, identical containers. For example, $S(7, 1) = 1$, $S(7, 3) = 301$. $S(m, n)$ is called the Stirling number of the second kind.

   Solution:

   a) $\sum_{i=1}^{5} \binom{5}{i}(i!)S(7, i) = \binom{5}{1}(1!)S(7, 1) + \binom{5}{2}(2!)S(7, 2) + \binom{5}{3}(3!)S(7, 3) + \binom{5}{4}(4!)S(7, 4) + \binom{5}{5}(5!)S(7, 5) = 78125 = 5^7$
   b) The expression m^n counts the number of ways to distribute n distinct objects among m distinct containers. For $1 \leq i \leq m$, let i count the number of distinct containers that we actually use – that is those that are not empty after the n distinct objects are

distributed. This number of distinct containers can be chosen in $\binom{m}{i}$ ways. Once we have the i distinct containers, we can distribute n distinct objects among i distinct containers, with no container left empty, in $(i!)S(n,i)$ ways. Therefore, we can interpret the expression $\sum_{i=1}^{m}\binom{m}{i}(i!)S(n,i)$ as counting the number of ways to distribute n distinct objects among m distinct containers. Then, we can conclude that $m^n = \sum_{i=1}^{m}\binom{m}{i}(i!)S(n,i)$

4) (1.5%) For distinct primes p, q let $A = \{p^m q^n \mid 0 \leq m \leq 31, 0 \leq n \leq 37\}$

  a) What is $|A|$?
  b) If $f : A \times A \rightarrow A$ is the closed binary operation defined by $f(a,b) = gcd(a,b)$, does f have an identity element?

Solution:

  a) $|A| = (32)(38) = 1216$
  b) The identity element is $p^{31}q^{37}$