Name: Student ID:

Quiz #4 5%

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This is a closed book test. Any academic dishonesty will automatically lead to zero point.

1) (1.5%) A sequence of numbers $a_1, a_2, \ldots$ is defined by $a_1 = 1,\ a_2 = 2,\ a_n = a_{n-1} + a_{n-2}, n \geq 3$
   a) Determine the value of a_5, a_6, a_7
   b) Prove that for all $n \geq 1, a_n < (\frac{7}{4})^n$

   Answer:
   a) $a_5 = 8, a_6 = 13, a_7 = 21$
   b) $a1 = 1 < (\frac{7}{4})^1$ so the result is true for $n = 1$, Likewise $a_2 = 2 < (\frac{7}{4})^2$ and the result holds for $n = 2$. Assume the result true for all $1 \leq n \leq k$, where $k > 2$. Now for $n = k+1$ we have $a_{k+1} = a_k + a_{k-1} < (\frac{7}{4})^k + (\frac{7}{4})^{k-1} = (\frac{7}{4})^{k-1}(\frac{11}{4}) = (\frac{7}{4})^{k-1}(\frac{44}{16}) < (\frac{7}{4})^{k-1}(\frac{49}{16}) = (\frac{7}{4})^{k-1}(\frac{7}{4})^2 = (\frac{7}{4})^{k+1}$ Hence, by the Principle of Mathematical Induction it follows that $a_n < (\frac{7}{4})^n$ for all $n \geq 1$.

2) (1.5%) Let $L_0, L_1, L_2 \ldots$ denote the Lucas numbers, where
   a) $L_0 = 2, L_1 = 1$
   b) $L_{n+2} = L_{n+1} + L_n$ for $n \geq 0$
   c) $L_1^2 + L_1^2 + L_3^2 + \ldots + L_n^2 = L_n L_{n+1} - 2,\ \forall n \in \mathbf{N}$

   If $n \in \mathbf{N}$, prove that $5F_{n+2} = L_{n+4} - L_n$, where F_n denotes the n^{th} Fibonacci number

   Answer:

   We adopt the Alternative Form of Principle of Mathematical Induction to prove $5F_{n+2} = L_{n+4} - L_n$.

   For $n = 0$, $5F_{0+2} = 5F_2 = 5(1) = 5 = 7 - 2 = L_4 - L_0 = L_{0+4} = L_0$

   For $n = 1$, $5F_{1+2} = 5F_3 = 5(2) = 10 = 11 - 1 = L_5 - L_1 = L_{1+4} = L_1$

   Next, we assume the induction hypothesis – that is, that for some $k \geq 1$, $5F_{n+2} = L_{n+4} - L_n$

for all $n = 0, 1, 2, \ldots, k-1, k$. It then follows that for $n = k+1$, $5F_{(k+1)+2} = 5F_{k+3} = 5(F_{k+2} + F_{k+1}) = 5(F_{k+2} + F_{(k-1)+2}) = 5F_{k+2} + 5F_{(k-1)+2} = (L_{k+4} - L_k) + (L_{(k-1)+4} - L_{(k-1)}) = (L_{k+4} - L_k) + (L_{k+3} - L_{k-1}) = (L_{k+4} + L_{k+3}) - (L_k + L_{k-1}) = L_{k+5} - L_{k+1} = L_{(k+1)+4} + L_{k+1}$. Hence, it then follows that $\forall n \in \mathbf{N}\ 5F_{n+2} = L_{n+4} - L_n$.

3) (1%) Show that for any $n \in \mathbf{Z}^+$, $gcd(5n+3, 7n+4) = 1$

Answer:

We find that for each $n \in \mathbf{Z}^+$, $(5n+3) \cdot (7) + (7n+4) \cdot (-5) = (35n+21) - (35n+20) = 1$. Hence, it follows that $5n+3$ and $7n+4$ are relatively prime.

4) (1%) Prove that $\sqrt{p}$ is irrational for any prime p. *Hint: Try proof by contradiction*

Answer:

If $\sqrt{p}$ is not irrational for any prime p, we have $\sqrt{p} = \frac{a}{b}$, where $a, b \in \mathbf{Z}^+$ and $gcd(a, b) = 1$. Then $\sqrt{p} = \frac{a}{b} \Rightarrow p = \frac{a^2}{b^2} \Rightarrow pb^2 = a^2 \Rightarrow p|a^2 \Rightarrow p|a$. We know that $a = pk\ \exists k \in \mathbf{Z}^+$, since $p|a$. Besides, $pb^2 = a^2 = (pk)^2$, or $b^2 = pk^2$. Hence, $p|b^2 \Rightarrow p|b$. However, if $p|a$ and $p|b$ then $gcd(a, b) = p > 1$. Here we can see that our conclusion $gcd(a, b) = p > 1$ contradicts our assumption $gcd(a, b) = 1$ in the very beginning.